Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 9, 1984, 125—131

# ON THE BOUNDARY BEHAVIOUR OF A ROTATION AUTOMORPHIC FUNCTION WITH FINITE SPHERICAL DIRICHLET INTEGRAL

RAUNO AULASKARI

In the paper [4] we defined a rotation automorphic function $f$ with respect to some Fuchsian group $\Gamma$. The function $f$, meromorphic in the unit disc $D$, was said to be rotation automorphic with respect to $\Gamma$ acting on $D$ if it satisfies the equation

$$f(T(z)) = S_T(f(z)), \tag{1}$$

where $T \in \Gamma$ and $S_T$ is a rotation of the Riemann sphere $\hat{C}$.

In [1]—[4] we supposed the rotation automorphic function $f$ to satisfy in a fundamental domain $F$ of $\Gamma$ the condition

$$\iint_F f^*(z)^2 \, d\sigma_z < \infty, \tag{2}$$

where $f^*(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of $f$ and $d\sigma_z$ the euclidean area element. Further, in [1], [2] and [4], we showed that, by suitable restrictions related to $F$, $f$ is a normal function in $D$, that is, $\sup_{z \in D}(1-|z|^2)f^*(z) < \infty$ (cf. [8]), while in [3] we constructed a non-normal rotation automorphic function $f$ satisfying the condition (2).

In this paper we shall consider the boundary behaviour of a rotation automorphic function $f$ satisfying (2). In fact, this work will be a continuation of the above-mentioned papers.

1. Let $D$ and $\partial D$ be the unit disc and the unit circle, respectively. We shall denote the hyperbolic distance by $d(z_1, z_2)(z_1, z_2 \in D)$ and the hyperbolic disc $\{z | d(z, z_0) < r\}$ by $U(z_0, r)$. Let $\chi(w_1, w_2)$ be the chordal distance between $w_1, w_2 \in \hat{C}$. We denote by $\Gamma$ a Fuchsian group acting on $D$ and by $\Omega$ the group of all Möbius transformations from $D$ onto itself.

The points $z, z' \in \bar{D} = D \cup \partial D$ are called $\Gamma$-equivalent if there exists a mapping $T \in \Gamma$ such that $z' = T(z)$. A domain $F \subset D$ is called a fundamental domain of $\Gamma$ if it does not contain two $\Gamma$-equivalent points and if every point in $D$ is $\Gamma$-equivalent to some point in the closure $\bar{F}$ of $F$. We fix the fundamental domain $F$ of $\Gamma$ to be a normal polygon in $D$. The point $\zeta \in \partial D$ is called a limit point of $\Gamma$ provided there is a point $z \in D$ and a sequence $(T_n)$ of different transformations of $\Gamma$ such that $T_n(z) \to \zeta$.

doi:10.5186/aasfm.1984.0902

All fixed points of parabolic transformations of $\Gamma$ are limit points and the number of such limit points is at most countable. Other limit points are called non-parabolic. By a hyperbolic ray we mean an arc $z\zeta$ of a circle orthogonal to $\partial D$ with an initial point $z\in D$ and $\zeta\in\partial D$. Let $\lambda$ be a hyperbolic ray. Each point of $\lambda$ has a $\Gamma$-equivalent point in $\bar{F}$. If the set of these points is everywhere dense in $F$, then $\lambda$ is said to be transitive (under $\Gamma$). A point $\zeta\in\partial D$ is called transitive if every hyperbolic ray through $\zeta$ is transitive.

1.1. Definition. The fundamental domain $F$ is called thick if there are positive constants $r, r'$ such that for any sequence of points $(z_n)\subset F$ there is a sequence of points $(z_n')$ for which $d(z_n, z_n')\leqq r$ and $U(z_n', r')\subset F$ for each $n=1, 2, \ldots$.

1.2. Remark. Suppose that the fundamental domain $F$ is thick. Let $s>0$ be fixed. Then, by the thickness of fundamental domains $T(F)$, $T\in\Gamma$, there is a positive integer $n(s)$, independent ot $z$, such that $U(z, s)$ has common points with at most $n(s)$ sets $T(\bar{F})$, $T\in\Gamma$.

We now suppose that $f$ is a rotation automorphic function in $D$. Let $K_0(f)$ be the set of points $\zeta\in\partial D$ such that

$$\lim_{z\to\zeta}(1-|z|^2)f^*(z) = 0$$

along each angular domain at $\zeta$ (cf. [10]). By an angular domain at $\zeta$ we mean a triangular domain whose vertices are $\zeta$ and two points of $D$. Let $K_+(f)$ be the set of points $\zeta\in\partial D$ such that

$$\limsup_{z\to\zeta}(1-|z|^2)f^*(z) > 0$$

along each angular domain at $\zeta$. Plainly, $K_0(f)\cap K_+(f)=\emptyset$. Let $F(f)$ be the set of all Fatou points [5, p. 21] of $f$.

1.3. Theorem. *Let $\Gamma$ be a finitely generated Fuchsian group and $f$ a non-constant rotation automorphic function with respect to $\Gamma$ satisfying the condition* (2). *If $L$ is the set of all non-parabolic limit points of $\Gamma$ on $\partial D$, then $K_0(f)=\partial D\setminus L$ and $K_+(f)=L$.*

*Proof.* Let $\zeta\in L$ and let $\Delta$ be an arbitrary angular domain at $\zeta$. Then, by [7, p. 181], we find a sequence of transformations $(T_n)\subset\Gamma$ and a sequence of points $(z_n)$ on the radius $0\zeta$ tending to $\zeta$ such that $z_n'=T_n(z_n)\in D(0, r)=\{z\,|\,|z|<r\}$, $r<1$, for each $n=1, 2, \ldots$. Since $f$ is non-constant, it is possible to choose a sequence of points $(w_k)\subset\Delta$ such that $w_k\to\zeta$, $(z_k)\subset(z_n)$, $d(z_k, w_k)\leqq R<\infty$, $w_k'=T_k(w_k)\to w_0'$ and $(1-|w_0'|^2)f^*(w_0')>0$. By the continuity of $f^*(z)$ we have

$$\lim_{k\to\infty}(1-|w_k|^2)f^*(w_k) = \lim_{k\to\infty}(1-|w_k'|^2)f^*(w_k')=(1-|w_0'|^2)f^*(w_0').$$

Hence $\zeta\in K_+(f)$ and thus $L\subset K_+(f)$.

By [1, 1.4. Theorem] $f$ is a normal function in $D$. Further, by a theorem of Pommerenke (cf. [9, Theorem 4]), $f$ has an angular limit at parabolic vertices. Let $P$ be the set of parabolic vertices. Since $F(f)\subset K_0(f)$ (cf. [10, Lemma 3]), $K_0(f)\supset P$. We denote by $C$ the set $\bigcup_{T\in\Gamma}\bigcup_{i=1}^{n} T(\bar{c}_i)$, where $\bar{c}_i$, $i=1, \dots, n$, are the closures of all free sides of $F$. Then, by [1, 1.2. Lemma], $(1-|z_m|^2)f^*(z_m)\to 0$ as $z_m\to\zeta\in T(\bar{c}_i)$, $i=1, \dots, n$; $T\in\Gamma$. Hence $C\subset K_0(f)$. For the disjoint sets $C$, $L$ and $P$ $\partial D=C\cup L\cup P$ holds. The equation $K_0(f)\cap K_+(f)=\emptyset$ implies that $K_0(f)\subset\partial D\setminus K_+(f)\subset\partial D\setminus L=C\cup P$ and $K_+(f)\subset\partial D\setminus K_0(f)\subset\partial D\setminus P\cup C=L$. The theorem follows.

1.4. Remark. If $\Gamma$ is of the first kind, then $C=\emptyset$. Since $P$ is at most countable, the linear Lebesgue measure of $K_0(f)$ is zero. Thus the linear Lebesgue measure of $K_+(f)$ is $2\pi$ (cf. [10, Theorem 2]).

1.5. Remark. Let $\Gamma$ be an arbitrary Fuchsian group (finitely or infinitely generated). We shall show that if the rotation automorphic function $f$ satisfies the condition (2), then $f$ has an angular limit at every parabolic vertex $p\in P$, that is, $K_0(f)\supset P$. Therefore let $P$ be a parabolic generator transformation fixing $p$ and let $f(P(z))=S_p(f(z))$. Let $S$ be a rotation of the Riemann sphere such that $(S\circ S_p)(z)=e^{i\varphi}S(z)$, where $\varphi\in\mathbf{R}$. We choose the transformations $T_n\in\Gamma$, $n=0, 1, \dots, n_0$, $T_0=id$, as follows: The fundamental domains $F, T_1(F), \dots, T_{n_0}(F)$, which have the common vertex $p$, are adjacent and $P$ maps a side $s$ of $F$ beginning at $p$ on a side $s'$ of $T_{n_0}(F)$. There is a fixed circle of $P$ passing through $p$ and cutting $s$ and $s'$. If $D_p$ denotes its interior, the set $\tau=\left(\bigcup_{n=0}^{n_0} T_n(\bar{F})\right)\cap D_p$ is called a parabolic sector at $p$. Let $1/(P(z)-p)=1/(z-p)+c$ and let the parameter mapping $t=e^{2\pi i/c(z-p)}$. If $g=S\circ f$, then $g(z(t))=t^\alpha h(z(t))$, where $h(z(t))$ is a meromorphic function in a parameter disc $\{t\,|\,|t|<\delta\}$ and $\alpha=\varphi/2\pi$. This follows from the condition (2) as shown in [1, 1.3. Lemma]. This implies $g$ to be continuous in the closure $\bar{\tau}$ where $\bar{\tau}$ has been taken in the closed unit disc $\bar{D}=\{z\,|\,|z|\le 1\}$. As $z\to p$ belonging to $s'$, we have $g(z)\to a$. On the other hand, $g(z)=g(P(w))=e^{i\varphi}g(w)\to e^{i\varphi}a$. Let $(z_n)$ be a sequence of points in an arbitrary angular domain $\Delta$ at $p$ tending to $p$. Then we can find the transformations $P^{m_n}$, $m_n\in Z$, such that $P^{m_n}(z_n)=z_n'\in\bar{\tau}$ and $z_n'\to p$ for $n\to\infty$. Hence

$$\chi(g(z_n), a) = \chi(e^{im_n\varphi}g(z_n), e^{im_n\varphi}a) = \chi(g(P^{m_n}(z_n)), a) = \chi(g(z_n'), a)\to 0$$

as $n\to\infty$. This implies that $g$ has an angular limit $a$ at $p$. Since $f=S^{-1}\circ g$, $f$ has an angular limit $S^{-1}(a)$ at $p$.

Next we shall prove the following lemma:

1.6. Lemma. *Let $f$ be a rotation automorphic function with respect to a Fuchsian group $\Gamma$. Suppose that the fundamental domain $F$ is thick and $\iint_F f^*(z)^2 d\sigma_z<\infty$. If $G_R=\{z\,|\,d(z, F)<R\}$ is any hull of $F$, then*

$$\iint_{G_R} f^*(z)^2\, d\sigma_z < \infty.$$

*Proof.* Since the fundamental domain $F$ is thick, there is, for every hyperbolic disc $U(z, R)$, $z\in\bar{F}$, an integer $n(R)$, depending only on the radius $R$, such that $U(z, R)$

intersects at most $n(R)$ fundamental domains $T(F)$, $T\in\Gamma$. Let $G_R=\{z|d(z, F)<R\}$ be any hull of $F$. We show that every point of $F$ has at most $n(R)$ $\Gamma$-equivalent points in $G_R$. Suppose, on the contrary, that there is a point $z_0\in F$ such that it has $n_1>n(R)$ $\Gamma$-equivalent points $z_i=T_i(z_0)$, $i=1, \dots, n_1$, in $G_R$. Then $U(z_0, R)$ intersects the fundamental domains $T_i^{-1}(F)$, $i=1, \dots, n_1$. Since $n_1>n(R)$, this is a contradiction and the assertion follows. Thus

$$\iint_{G_R} f^*(z)^2\, d\sigma_z \leqq n(R)\iint_F f^*(z)^2\, d\sigma_z < \infty,$$

and the lemma is proved.

By using the above lemma we obtain

1.7. Theorem. *Let $f$ be a rotation automorphic function with respect to $\Gamma$. Suppose that the fundamental domain $F$ is thick and $\iint_F f^*(z)^2\, d\sigma_z<\infty$. Then, for each sequence of points $(z_n)\subset G_R=\{z|d(z, F)<R\}$ converging to $\partial D$,*

$$\lim_{n\to\infty}(1-|z_n|^2)f^*(z_n)=0$$

*holds.*

*Proof.* We choose a sequence of points $(z_n)\subset G_R$ tending to $\partial D$ and the hyperbolic discs $U(z_n, R)\subset G_{2R}$, $n=1, 2, \dots$. By 1.6. Lemma

$$S_n(R)=\frac{1}{\pi}\iint_{U(z_n, R)} f^*(z)^2\, d\sigma_z \to 0 \tag{1.1}$$

as $n\to\infty$. Define the transformations

$$z=V_n(\zeta)=\frac{\zeta+z_n}{1+\bar{z}_n\zeta}$$

and the functions

$$f_n(\zeta)=f(V_n(\zeta)).$$

By [6, Theorem 6.1] we have

$$f_n^*(0)^2 \leqq \frac{1}{x^2}\frac{S_n(R)}{1-S_n(R)}, \tag{1.2}$$

where $x=(e^{2R}-1)/(e^{2R}+1)$. On the other hand,

$$f_n^*(0)=(1-|z_n|^2)f^*(z_n). \tag{1.3}$$

By (1.1), (1.2) and (1.3),

$$\lim_{n\to\infty}(1-|z_n|^2)f^*(z_n)=0,$$

which is the assertion.

1.8. Remark. This result improves Theorem 5 of [4] according to which $f$, satisfying (2), is a normal function in $D$.

In what follows $\bar{G}_R$ denotes the closure of $G_R$ taken in the closed unit disc $\bar{D}=\{z|\,|z|\leqq 1\}$.

1.9. Corollary. *By the assumptions of* 1.7. *Theorem every point* $\zeta \in \bar{G}_R \cap \partial D$ *belongs to* $K_0(f)$.

*Proof.* Let $\zeta \in \bar{G}_R \cap \partial D$ be any point and $\Delta$ an arbitrary angular domain at $\zeta$. We find a positive real number $R$ such that $G_R \supset \Delta$. Hence $\zeta \in K_0(f)$.

1.10. Remark. In the proof of 1.3. Theorem we obtained $K_+(f) \supset L$ supposing that $f$ is only a rotation automorphic function, that is, $f$ need not satisfy the integral condition (2). This result, which is valid for a finitely generated Fuchsian group, generalizes 2.4. Theorem in [2].

Again we assume that $f$ is a non-constant rotation automorphic function satisfying (2). Let $F$ be the Riemannian image of $D$ by $f$ covering $\hat{C}$. Let $\gamma(z, f)$ be the maximum of $q$, $0 < q \leq 1$, such that $F$ contains the schlicht disk $\{w \in \hat{C} | \chi(w, f(z)) < q\}$ of centre $f(z) \in F$; if $f^*(z) = 0$, we set $\gamma(z, f) = 0$ (cf. [10, p. 143]). Let $Q_0(f)$ be the set of points $\zeta \in \partial D$ such that

$$\lim_{z \to \zeta} \gamma(z, f) = 0$$

along each angular domain at $\zeta$. Let $Q_+(f)$ be the set of points $\zeta \in \partial D$ such that

$$\limsup_{z \to \zeta} \gamma(z, f) > 0$$

along each angular domain at $\zeta$. By applying [10, Lemma 4] in Theorem 1.3 we obtain $Q_0(f) = \partial D \setminus L$ and $Q_+(f) = L$ provided, $\Gamma$ is finitely generated.

2. In what follows we shall get rid of the assumption that a rotation automorphic function satisfies the integral condition (2). Let $f$ be a non-constant rotation automorphic function with respect to $\Gamma$, $\zeta$ a hyperbolic fixed point of $\Gamma$ and $\xi$ a transitive point of $\Gamma$. In [2, 2.3. Theorem, 2.7. Theorem] we proved that $\zeta \notin F(f)$ and $\xi \notin F(f)$. We now show that $\zeta \in K_+(f)$ and $\xi \in K_+(f)$.

We shall first prove $\zeta \in K_+(f)$, where $\zeta$ is a hyperbolic fixed point of $T$, $T \in \Gamma$. Choose a circle through $\zeta$ and the other fixed point $\zeta'$ of $T$ and denote its arc lying in $D$ by $C$. Further, we choose a point $z_0$ on $C$ such that $f^*(z_0) > 0$. Now $\zeta$ is an attractive fixed point of either $T$ or $T^{-1}$ and we suppose that $\zeta$ is an attractive fixed point, then $z_n = T^n(z_0) \to \zeta$ as $n \to \infty$. Since $(1 - |z_0|^2) f^*(z_0) = (1 - |z_n|^2) f^*(z_n)$ and an arbitrary angular domain $\Delta$ contains the end of some circle $C$, $\zeta \in K_+(f)$.

The reasoning in the case of a transitive point $\xi$ is the following: If $\Delta$ is any angular domain at $\xi$, we find a sequence of $(T_n) \subset \Gamma$ and a point $z_0 \in D$ such that $(z_n) = (T_n(z_0)) \subset \Delta$, $z_n \to \xi$ and $f^*(z_0) > 0$. Since $(1 - |z_0|^2) f^*(z_0) = (1 - |z_n|^2) f^*(z_n)$, $\xi \in K_+(f)$.

2.1. Remark. Let $\Gamma$ be of divergence type, that is, $\sum_{T \in \Gamma} (1 - |T(z)|^2) = \infty$, $z \in D$. Then the set of all transitive points has the linear Lebesgue measure equal to $2\pi$. Since, by the above, the set of all transitive points belongs to $K_+(f)$, the linear Lebesgue measure of $K_+(f)$ is $2\pi$. This implies that the linear Lebesgue measure of $K_0(f)$ is zero (cf. 1.4. Remark).

A rotation automorphic function $f$ is said to be of the second kind if there exists a sequence of points $(z_n)$ in the closure $\bar{F}$ such that the sequence of functions

$$f_n(\zeta) = f\left(\frac{\zeta+z_n}{1+\bar{z}_n\zeta}\right)$$

tends uniformly to a constant limit in some neighbourhood of $\zeta=0$.

We proved in [2, 2.2. Theorem] the following theorem: Let $f$ be a rotation automorphic function with respect to $\Gamma$. If $F(f)\neq\emptyset$, then $f$ is of the second kind.

Now we shall improve this result as follows:

2.2. Theorem. *Let $f$ be a rotation automorphic function with respect to $\Gamma$. If $K_0(f)\neq\emptyset$, then $f$ is of the second kind.*

*Proof.* Suppose that $\xi_0\in K_0(f)$. Let $(z_n)\subset\Delta$ ($\Delta$ an angular domain at $\xi_0$) be a sequence of points converging to $\xi_0$. We choose the transformations $L_n\in\Omega$, $T_n\in\Gamma$ such that $L_n(0)=z_n$, $T_n(z_n)=z_n'\in\bar{F}$ and $(T_n\circ L_n)(\zeta)=(\zeta+z_n')/(1+\bar{z}_n'\zeta)$ for each $\zeta\in D$ and $n=1, 2, \ldots$.

Define the functions

$$g_n(\zeta) = f(L_n(\zeta)).$$

By a small computation we obtain

$$g_n^*(\zeta) = \frac{1}{1-|\zeta|^2}\cdot(1-|L_n(\zeta)|^2)f^*(L_n(\zeta)).$$

Then $(L_n(\zeta))$ belongs to an angular domain $\Delta'$ at $\xi_0$ and converges to $\xi_0$ for all $\zeta\in U(0,r)$, $r<\infty$, as $n\to\infty$. Hence, by the assumption, $(1-|L_n(\zeta)|^2)f^*(L_n(\zeta))\to 0$ and thus $g_n^*(\zeta)\to 0$ as $n\to\infty$. This implies that

$$g_n^*(\zeta) \leq M < \infty$$

for each $\zeta\in U(0,r)$ and $n=1, 2, \ldots$. Therefore $\{g_n\}$ is a normal family in $D$ by Marty's criterion. Now we find a subsequence $(g_k)$ of $(g_n)$ converging to $g_0$ uniformly on every compact part of $D$. Thus

$$g_0^*(\zeta) = \lim_{k\to\infty} g_k^*(\zeta) = 0$$

uniformly in $U(0,r)$. As a meromorphic function $g_0(\zeta)\equiv c$ (a constant). By continuing in the same way as in [2, the proof of 2.2.Theorem] we find a subsequence $(h_m)$ of $(h_k)=(f\circ T_k\circ L_k)$ converging uniformly to a constant in $U(0,r)$. The theorem follows.

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University of Joensuu
Department of Mathematics
SF—80101 Joensuu 10
Finland

Received 16 January 1984