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THE L^2-COHOMOLOGY OF NEGATIVELY CURVED RIEMANNIAN SYMMETRIC SPACES

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Let G be a connected linear semi-simple Lie group, K a maximal compact sub group of G. As is well-known, the quotient space $X=G/K$ is homeomorphic to euclidean space and, endowed with a G-invariant metric, is a Riemannian symmetric space with negative curvature without flat component and any such space can be obtained in this way. We fix an irreducible finite dimensional representation (r, E) of G. Our object of interest in this paper is the L^2-cohomology space $H^{\cdot}_{(2)}(X; E)$ of X with respect to E. It can be defined first as the cohomology of the complex $A^{\cdot}_{(2)}(X; E)$ of E valued smooth differential forms η on X such that η and $d\eta$ are square integrable, where d is exterior differentiation. To get a Hilbert space definition, we may consider the completion $\bar{A}^{\cdot}_{(2)}(X; E)$ of $A^{\cdot}_{(2)}(X; E)$ with respect to the square norm $(\eta, \eta)+(d\eta, d\eta)$, and the graph closure or strong closure $\bar{d}$ of d. It is known that the inclusion $A^{\cdot}_{(2)}(X; E)\to\bar{A}^{\cdot}_{(2)}(X; E)$ induces an isomorphism in cohomology [6]. The group G operates on these complexes and hence on the cohomology. In the Hilbert space definition, $H^{\cdot}_{(2)}(X; E)$ appears as the quotient of the closed subspace of the cocycles in $\bar{A}_{(2)}(X; E)$ by the image of $\bar{d}$. Therefore, if $\bar{d}$ has a closed range, then $H^{\cdot}_{(2)}(X; E)$ has a natural Hilbert space structure and yields a unitary representation of G. Our first objective is to prove that this is the case when G and K have the same rank and to identify the representations thus obtained. We shall prove

Theorem A. *Let* $m=(\dim X)/2$ *and assume that* $\operatorname{rk} G=\operatorname{rk} K$. *Then*
(i) *The range of* $\bar{d}$ *is closed. We have*

(1) $$H^i_{(2)}(X; E) = 0, \quad \text{if} \quad i \neq m.$$

(ii) *The* G*-space* $H^m_{(2)}(X; E)$ *is the direct sum of the discrete series representations of* G *having the same infinitesimal character as* (r, E).

The proof of (ii) shows in fact that $H^m_{(2)}(X; E)$ may be identified with the space of square integrable harmonic m-forms. Interpreted in this way, (ii) is quite reminiscent of some characterizations of the discrete series as spaces of harmonic square integrable sections of certain K-bundles over X (see e.g. [10]).

We shall also obtain some information in the case of unequal ranks:

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Theorem B. *Assume that* $l_0 = \mathrm{rk}\, G - \mathrm{rk}\, K$ *is not zero.*

(i) *If* E *is not equivalent to its contragredient complex conjugate* $\bar{E}^*$, *then* $H^{\cdot}_{(2)}(X; E) = 0$.

(ii) *If* $E \sim \bar{E}^*$, *then* $\bar{d}$ *does not have closed range, and* $H^i_{(2)}(X, E)$ *is infinite dimensional at least for* $i \in (m - (l_0/2), m + (l_0/2)]$.

Our starting point is a regularization theorem of [1] which yields a canonical isomorphism

$$\mathrm{Ext}^{\cdot}_{(\mathfrak{g}, K)}(E^*, L^2(G)^{\infty}) \tilde{\to} H^{\cdot}_{(2)}(X; E), \tag{1}$$

where the left-hand side refers to Ext$^{\cdot}$ in the category of $(\mathfrak{g}, K)$-modules (cf. [5:I]) and $L^2(G)$ is viewed as a G-module via the right regular representation. We may then investigate the left-hand side using the results of Harish-Chandra [8] on $L^2(G)$. This reduces us to the computation of $\mathrm{Ext}^{\cdot}_{(\mathfrak{g}, K)}(E^*, L^{\infty}_{P,\omega})$, where the $L_{P,\omega}$ are the direct summands of $L^2(G)$ given by [8]. Those are defined in Section 1, and the computations performed in Section 2. Theorems A and B are proved in Section 3.

This procedure is quite similar to the study of $H^{\cdot}_{(2)}(\Gamma \backslash X; E)$ in [2], where Γ is a discrete subgroup of finite covolume of G. In fact, (1) above is also valid if X and G are replaced by $\Gamma \backslash X$ and $\Gamma \backslash G$ (for any discrete $\Gamma \subset G$). Modulo a result of [3] (whose role is played here by 1.4), we are again reduced to the discussion of Ext$^{\cdot}$-groups with respect to some elementary subspaces of $L^2(\Gamma \backslash G)$ which are given by Langlands' results [11]. In short, [2] and the present paper correspond to the two cases where extensive information on $L^2(\Gamma \backslash G)$ is available.

Some notation. The Lie algebra of a Lie group $A, G, \ldots$ is denoted by the corresponding lower case German letter $\mathfrak{a}, \mathfrak{g}, \ldots$.

A reductive group is always meant to satisfy the conditions of [5: 0, 3.1]. In particular, it belongs to Harish-Chandra's class [7].

The space of smooth vectors of a continuous representation (π, V) of a Lie group L is denoted V^{∞}. If the center $\mathscr{Z}$ of the universal enveloping algebra of L acts by scalars on V^{∞}, we denote by χ_π or χ_V the character of $\mathscr{Z}$ thus obtained, the so-called infinitesimal character of π.

The contragredient of a representation (π, V) is denoted (π^*, V^*).

The set of equivalence classes of irreducible unitary (resp. square integrable) representations of the reductive group L with compact center is denoted $\hat{L}$ (resp. $\hat{L}_d$). If L is compact and F a finite subset of L, then, for any continuous L-module V, we let V_F denote the sum of the isotypic subspaces V_τ $(\tau \in F)$.

1. The decomposition of $L^2(G)$

In this section, we recall some of the fundamental results of Harish-Chandra [8] on the spectral decomposition of $L^2(G)$, in a form adapted to our needs.

1.1. Let (P, A) be a p-pair (cf. [7] or [5: 0, 3.4]) and $P=N_P A_P M_P$ or simply $P=NAM$ the associated Langlands decomposition of P. In particular, N is the unipotent radical of P, A is a split component of the radical of P and the centralizer $Z(A)$ of A in G is the direct product of A and M. For $\lambda\in\mathfrak{a}_c^*$, we denote by $\boldsymbol{C}_\lambda$ the one-dimensional representation of A, where $a\in A$ acts by multiplication by $a^\lambda=\exp\lambda(\log a)$. Given $(\omega, V_\omega)\in\hat{M}_d$ and $\lambda\in\mathfrak{a}_c^*$, we view as usual $V_\omega\otimes\boldsymbol{C}_\lambda$ as a representation of P on which N acts trivially. Let

$$(1)\qquad I_{P,\omega,i\mu}=\operatorname{Ind}_P^G(V_\omega\otimes\boldsymbol{C}_{\varrho_P+i\mu})\quad(\omega\in\hat{M}_d;\ \mu\in\mathfrak{a}_c^*)$$

where ϱ_P is defined by

$$a^{2\varrho_P}=\det\operatorname{Ad}a|_{\mathfrak{n}}\quad(a\in A).$$

It is unitary if $\mu\in\mathfrak{a}^*$, our only case of interest in this paper.

We shall assume that A and M are stable under the Cartan involution of G associated to K. In particular, $K\cap M$ is a maximal compact subgroup of M and P. We recall that the K-module structure of $I_{P,\omega,i\mu}$ is "independent of μ", i.e., there exists a canonical K-equivariant isomorphism of Hilbert space of $I_{P,\omega,i\mu}$ on a fixed K-module $U_{(\omega)}$, namely $U_{(\omega)}=\operatorname{Ind}_{K\cap M}^K(V_\omega)$, where V_ω is viewed as a $(K\cap M)$-module. In particular, the K-types of $I_{P,\omega,i\mu}$ are independent of μ.

1.2. Recall that a parabolic subgroup P is *cuspidal* if M has the same rank as its maximal compact subgroups, *fundamental* if M contains a Cartan subgroup of K. The group G is its own fundamental parabolic subgroup if and only if G and K have the same rank. According to [8], there exists a finite set S of non-conjugate cuspidal parabolic subgroups of G, containing exactly one fundamental parabolic subgroup, with the following properties:

$$(1)\qquad L^2(G)=\tilde{\bigoplus}_{P\in S}L_P,$$

with

$$(2)\qquad L_P=\tilde{\bigoplus}_{\omega\in\hat{M}_{P,d}}L_{P,\omega},$$

$$(3)\qquad L_{P,\omega}=\int_{\mathfrak{a}^*}^{\oplus}I_{P,\omega,i\mu}^*\hat{\otimes}I_{P,\omega,i\mu}\,d\mu_\omega.$$

Here $d\mu_\omega$ is a certain measure (the Plancherel measure), which is the product of an analytic function by the Lebesgue measure, $\tilde{\oplus}$ stands for a Hilbert direct sum and $\hat{\otimes}$ for the usual Hilbert space completion of the algebraic tensor product of two Hilbert spaces.

The action of G by left (resp. right) translations is given by the natural action in (3) on the first (resp. second) factor of the integrand. If $P=G$, then (3) can be written more simply as

$$(4) \qquad L_{P,\omega} = V_\omega^* \hat{\otimes} V_\omega d_\omega,$$

where ω runs through $\hat{G}_d$ and d_ω is the formal degree of ω.

The spaces $L_{P,\omega}$ will be called *elementary subspaces of* $L^2(G)$.

1.3. *Casimir operators.* We fix an admissible trace form on $\mathfrak{g}$ [3: 2.3], say the Killing form if $\mathfrak{g}$ is semi-simple, and for a reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$, denote by $C_\mathfrak{m}$ the Casimir operator associated to the same trace form. If $C_\mathfrak{m}$ acts by a scalar multiple of the identity on the space H^∞ of smooth vectors of a continuous representation (π, H_π) of a reductive subgroup M of G with Lie algebra $\mathfrak{m}$, we denote by $c(\pi)$ or $c(H_\pi)$ the eigenvalue of $C_\mathfrak{m}$. We recall that $C_\mathfrak{g}$ acts by a scalar multiple of the identity on $I^\infty_{P,\omega,i\mu}$ and that there exists a constant e_P, depending only on G and P, such that

$$(1) \qquad c(I_{P,\omega,i\mu}) = e_P + c_\mathfrak{m}(V_\omega) - (\mu, \mu) \quad (\omega \in \hat{M}_d;\ \mu \in \mathfrak{a}^*).$$

1.4. Lemma. *Let J be a finite subset of $\hat{K}$. Then we can write $L^2(G)$ as a direct sum of two G-stable subspaces Q, R such that Q is the sum of finitely many elementary subspaces and $R_J = 0$.*

By 1.2(1), it suffices to prove the existence of such a decomposition for a space L_P ($P \in S$). Let J_P be the set of $(K \cap M)$-types occurring in the restriction to $K \cap M$ of the elements $\tau \in J$. It is finite. Let $U_{(\omega)}$ be as in 1.1. By Frobenius reciprocity, $U_{(\omega),J} \neq 0$ implies $V_{\omega,J_P} \neq 0$. It is known that there are only a finite number of $\omega \in \hat{M}_d$ containing a given $(K \cap M)$-type: this follows from the description of the $(K \cap M)$-types in a discrete series representation given by Blattner's formula [9], which in particular shows the existence of a single minimal $(K \cap M)$-type with multiplicity one. As a consequence the set of $L_{P,\omega}$ with a non-trivial J-component is finite. We let then Q be their direct sum and R the orthogonal complement of Q in L_P.

2. Relative Lie algebra cohomology with respect to an elementary subspace

2.1. We consider in this section the cohomology space $\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L^\infty_{P,\omega})$, where $L_{P,\omega}$ is viewed as a G-module via right translations (1.2). It is therefore the cohomology of the complex

$$(1) \quad C^{\cdot}(\mathfrak{g}, K; L^\infty_{P,\omega} \otimes E) = \mathrm{Hom}_K(\Lambda^{\cdot}\mathfrak{g}/\mathfrak{k}, L^\infty_{P,\omega} \otimes E) = \mathrm{Hom}_K(\Lambda^{\cdot}\mathfrak{g}/\mathfrak{k} \otimes E^*, L^\infty_{P,\omega}).$$

If J is the (finite) set of K-types occurring in $\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k}) \otimes E^*$, we have therefore

$$(2) \quad C^{\cdot}(\mathfrak{g}, K; L^\infty_{P,\omega} \otimes E) \subset \mathrm{Hom}_K(\Lambda^{\cdot}\mathfrak{g}/\mathfrak{k} \otimes E^*, L^\infty_{P,\omega,J}) = \mathrm{Hom}_K(\Lambda^{\cdot}\mathfrak{g}/\mathfrak{k}, L^\infty_{P,\omega,J} \otimes E).$$

The action of G by left translations is an automorphism of this complex and goes over to the cohomology. This complex is contained in the graded Hilbert space

$\mathrm{Hom}_K(\Lambda^* \mathfrak{g}/\mathfrak{k}, L_{P,\omega} \otimes E)$ (where, as usual, E is endowed with an "admissible" scalar product, i.e., one which is invariant under K and with respect to which the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ is represented by self-adjoint operators). If $\bar{d}$ is the closure of d, then the inclusion $C^{\cdot}(\mathfrak{g}, K; L^{\infty}_{P,\omega} \otimes E) \to \mathrm{dom}\, \bar{d}$ is an isomorphism in cohomology [1: 2.7].

2.2. We first consider the case of a discrete series representation, i.e., where $P=G$ and $L_{P,\omega} = V^*_\omega \hat{\otimes} V_\omega$, with $\omega \in \hat{G}_d$. Since $V_{\omega,J}$ is finite dimensional and consists of smooth vectors we have then

$$(V^*_\omega \hat{\otimes} V_\omega)^{\infty}_J = V^*_\omega \otimes V_{\omega,J},$$

(note that $\hat{\otimes}$ has been replaced by $\otimes$), whence

$$C^{\cdot}(\mathfrak{g}, K; L^{\infty}_{P,\omega} \otimes E) = V^*_\omega \otimes C^{\cdot}(\mathfrak{g}, K; V^{\infty}_\omega \otimes E).$$

Since the second factor on the right-hand side is finite dimensional, we see that $d = \bar{d}$ and that

$$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega}) = V^*_\omega \otimes \mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, V^{\infty}_\omega),$$

this isomorphism being G-equivariant, G operating through the given representation on the first factor of the right-hand side, and trivially on the second factor. But the value of the second factor is well-known [5: II, 5.3] (see 2.9), therefore we get

2.3. Proposition. *Let* $P=G$ *and* $\omega \in \hat{G}_d$. *Then* $\bar{d}$ *has closed range. We have* $\mathrm{Ext}^i_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega}) = 0$ *if* $i \neq m = (\dim X)/2$ *or* $\chi_\omega \neq \chi_{r^*}$. *If* $\chi_\omega = \chi_{r^*}$, *then*

(1)
$$\mathrm{Ext}^m_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega}) = V^*_\omega.$$

2.4. Assume now that $P \neq G$, hence that $L_{P,\omega}$ is a direct integral of induced representations. We have

(1)
$$L^{\infty}_{P,\omega,J} = \Big(\int^{\oplus}_{\mathfrak{a}^*} (I^*_{P,\omega,i\mu} \hat{\otimes} I_{P,\omega,i\mu,J})\, d\mu_\omega\Big)^{\infty}.$$

But $I_{P,\omega,i\mu,J}$ is finite dimensional, so that we may again replace $\hat{\otimes}$ by $\otimes$ and write

(2)
$$L^{\infty}_{P,\omega,J} = \Big(\int^{\oplus}_{\mathfrak{a}^*} I^*_{P,\omega,i\mu} \otimes I_{P,\omega,i\mu,J}\, d\mu_\omega\Big)^{\infty}.$$

2.5. We shall have to use some results of [5: III] on cohomology with respect to $I_{P,\omega,i\mu}$. We recall them here, together with some of the relevant notation. We fix a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$, let $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$, fix an ordering on the set $\Phi(\mathfrak{g}_c, \mathfrak{h}_c)$ of roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$ compatible with $\Phi(P, A)$ and let W^P be the usual canonical set of representatives of right classes of the Weyl group $W(\mathfrak{g}_c, \mathfrak{h}_c)$ of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$ modulo the Weyl group $W(\mathfrak{m}_c \oplus \mathfrak{a}_c, \mathfrak{h}_c)$ of $\mathfrak{m}_c \oplus \mathfrak{a}_c$ with respect to $\mathfrak{h}_c$. Furthermore let $\lambda - \varrho$ be the highest weight of r, where $\lambda \in \mathfrak{h}^*_c$ is dominant and 2ϱ is the sum of the positive roots. Since μ is real, [5: III, 3.3] shows that for

$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, I^{\infty}_{P,\omega,i\mu})$ not to be zero, first there must exist $s \in W^P$ such that

(1) $$s(\lambda)|_{\mathfrak{a}^*} = 0, \ \chi_{-s(\lambda)|\mathfrak{b}_c} = \chi_\omega;$$

this condition is independent of μ and satisfied by at most one $s \in W^P$. Furthermore, we must also have

(2) $$\mu = 0.$$

By assumption, P is cuspidal, hence for the first equality of (1) to hold, it is necessary that P be fundamental [5: III, 5.1].

2.6. Lemma. *Assume that* $\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, I^{\infty}_{P,\omega,0})=0$. *Then*

(1) $$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega}) = 0.$$

Recall that the K-types of $I^{\infty}_{P,\omega,i\mu}$ are independent of μ. If $I_{P,\omega,i\mu,J}=0$ for some μ, then it is so for all μ's and $C^{\cdot}(\mathfrak{g}, K; L_{P,\omega}\otimes E)=0$, which obviously yields (1). Assume now that $I_{P,\omega,i\mu,J}\neq 0$, hence that

(2) $$C^{\cdot}(\mathfrak{g}, K; I^{\infty}_{P,\omega,i\mu}\otimes E) \neq 0 \quad (\mu\in\mathfrak{a}^*).$$

In view of the assumption of 2.6 and of the results recalled in 2.5, we have

(3) $$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, I^{\infty}_{P,\omega,i\mu}) = 0 \quad (\mu\in\mathfrak{a}^*).$$

By [5: II, 3.1], we deduce from (2) and (3):

(4) $$c(I_{P,\omega,i\mu})-c(E) \neq 0 \quad (\mu\in\mathfrak{a}^*).$$

It follows from 1.3(1) that, given a constant $d>0$, there exists a compact set $D\subset\mathfrak{a}^*$ such that $|c(I_{P,\omega,i\mu})-c(E)|\geqq d$ outside D. Since $c(I_{P,\omega,\lambda})$ is a continuous function of λ, we see that there exists $c>0$ such that

(5) $$|c(I_{P,\omega,i\mu})-c(E)| \geqq c \quad \text{for all} \quad \mu\in\mathfrak{a}^*.$$

The constant $c(I_{P,\omega,i\mu})$ is also the eigenvalue of the Casimir operator on $(I^*_{P,\omega,i\mu}\hat{\otimes} I_{P,\omega,i\mu})^\infty$, it being understood that G acts only on the second factor. The Casimir operator $C_\mathfrak{g}$ operates therefore on $L^{\infty}_{P,\omega}$ by the rule

$$C_\mathfrak{g} f(\mu) = c(I_{P,\omega,i\mu})f(\mu), \quad (f\in L^{\infty}_{P,\omega};\, \mu\in\mathfrak{a}^*).$$

From (5), we see then that $(C_\mathfrak{g}-c(E)\cdot I)$ has a bounded inverse on $L^{\infty}_{P,\omega}$. Therefore 2.5 follows from 5.2 in [3].

2.7. Proposition. *Assume that P is not fundamental. Then*

$$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega})=0.$$

In fact, as recalled in 2.5, the assumption of 2.6 is satisfied if P is not fundamental.

2.8. Proposition. *Assume P to be fundamental. Then either*

$$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega})=0$$

or $\bar{d}$ does not have closed range.

If 2.5(1) is not fulfilled, then $\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega})=0$ by 2.6. Assume then 2.5(1) to hold. Let

(1) $$L'_{P,\omega} = \int^{\oplus}_{\mathfrak{a}^*} I_{P,\omega,i\mu}\, d\mu_\omega .$$

As recalled in 1.1, there is a canonical K-isomorphism

(2) $$\alpha: I_{P,\omega,i\mu} \xrightarrow{\sim} U = \mathrm{Ind}^K_{K\cap M}(V_\omega) \quad (\mu\in\mathfrak{a}^*).$$

From this we get a canonical injective $(K\times G)$-homomorphism

(3) $$U_{(K)}\otimes L'_{P,\omega} \to L_{P,\omega}$$

where $U_{(K)}$ is the space of K-finite vectors in U, and a K-equivariant homomorphism

(4) $$\beta\colon \mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, U_{(K)}\otimes L'^{\infty}_{P,\omega}) = U_{(K)}\otimes \mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L'^{\infty}_{P,\omega}) \to \mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega}).$$

We claim that β is injective. For any $\tau\in\hat{K}$, the space U_τ is finite dimensional. Let us denote by ${}_\tau L_{P,\sigma}$ the isotypic component of type τ for the left action of K, i.e., on the factors $I^*_{P,\omega,i\mu}$. It is clear from the definitions that α induces a $(K\times G)$-isomorphism of $U_\tau\otimes L'_{P,\omega}$ onto ${}_\tau L_{P,\omega}$. We have an isomorphism

$$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, U_{(K)}\otimes L'^{\infty}_{P,\omega}) = \oplus_{\tau\in\hat{K}}\, \mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, U_\tau\otimes L'_{P,\omega}).$$

Let $\eta\in C^{\cdot}(\mathfrak{g}, K; U_{(K)}\otimes L'^{\infty}_{P,\omega}\otimes E)$ be a cocycle. We may write

$$\eta = \textstyle\sum_{\tau\in\hat{K}}\eta_\tau,$$

where η_τ is a cocycle in $C^{\cdot}(\mathfrak{g}, K; U_\tau\otimes L'^{\infty}_{P,\omega}\otimes E)$. Let F be the set of τ's for which $\eta_\tau\neq 0$. It is finite. Let $\varphi_\tau\in C^\infty_c(K)$ be the function which defines the projector of any continuous K-module onto its τ-isotypic component and let $\varphi_F=\sum_{\tau\in F}\varphi_\tau$. Assume now that $\alpha(\eta)=d\mu$ for some $\mu\in C^{\cdot}(\mathfrak{g}, K; L^{\infty}_{P,\omega}\otimes E)$. Then we have

$$\alpha_F * \eta = \eta = d(\alpha_F * \mu),$$

since the operation of K on the left commutes with differentiation. The element $\alpha_F*\mu$ is contained in $C^{\cdot}(\mathfrak{g}, K; {}_F L^{\infty}_{P,\omega}\otimes E)$, which can be identified to the image under α of $C^{\cdot}(\mathfrak{g}, K; U_F\otimes L'^{\infty}_{P,\omega}\otimes E)$. Therefore η is already cohomologous to zero in the latter space, which proves our contention.

For any finite dimensional subspace W of $U_{(K)}$ we have

(5) $$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, W\otimes L'^{\infty}_{P,\omega}) = W\otimes \mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L'^{\infty}_{P,\omega}).$$

By the same proof as that of 3.4 in [2], one shows:

(6) $$\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L'^{\infty}_{P,\omega}) = \Big(\mathrm{Ext}^{\cdot}_{(\mathfrak{m},K\cap M)}(F^*_{s\lambda|\mathfrak{b}_c}, V^{\infty}_{\omega})\otimes H^{\cdot}\Big(\mathfrak{a}, \int^{\oplus}_{\mathfrak{a}^*} C_{i\mu}\, d\mu_\omega\Big)\Big)[-(\dim N)/2]$$

where $\lambda \in \mathfrak{a}_c^*$ is dominant such that $\chi_\lambda = \chi_r$. [The only difference is that $d\mu_\omega$ replaces the Lebesgue measure, but this does not affect the argument.] Since we assume the left-hand side to be non-zero, the first factor of the right-hand side is not zero. We claim that, as in [2: 3.2], we have

$$H^0\left(\mathfrak{a};\ \int_{\mathfrak{a}^*}^{\oplus} C_{i\mu}\, d\mu_\omega\right) = 0, \tag{7}$$

$$\dim H^i\left(\mathfrak{a};\ \int_{\mathfrak{a}^*}^{\oplus} C_{i\mu}\, d\mu_\omega\right) = \infty \quad (i = 1, \dots, \dim \mathfrak{a}),\ \bar{d} \textit{ is not closed.} \tag{8}$$

The group P is fundamental, therefore $d\mu_\omega$ is the product of the Lebesgue measure $d\mu$ by a polynomial, say R, which is strictly positive on the regular elements [8: § 24, Theorem 1]. From this (7) follows as in *loc. cit.* As regards (8), it is enough to prove it if the direct integral is taken over some measurable set D of strictly positive measure. Take for instance for D the positive Weyl chamber. Let $R^{1/2}$ be the positive square root of R on D. Then $\varphi \mapsto R^{1/2}\varphi$ defines an equivariant isomorphism

$$\int_D^{\oplus} C_{i\mu}\, d\mu_\omega \xrightarrow{\sim} \int_D^{\oplus} C_{i\mu}\, d\mu \tag{9}$$

which reduces us to [2: 3.2]. Since the map β of (4) is injective, 2.8 follows.

Remarks. (1) The first factor on the right-hand side of (6) is non-zero only in the middle dimension $m_0 = (\dim M/(K_M \cap M))/2$ [5: II, 5.3]. Let moreover $l_0 = \dim A$ (i.e., $l_0 = \operatorname{rk} G - \operatorname{rk} K$). Since $2m = 2m_0 + l_0 + \dim N$, we get:

$$\dim \operatorname{Ext}^i_{(\mathfrak{g}, K)}(E^*, L'^{\infty}_{P,\omega}) = \begin{cases} \infty & i \in (m-(l_0/2),\ m+(l_0/2)] \\ 0 & i \notin (m-(l_0/2),\ m+(l_0/2)], \end{cases}$$

assuming that $\operatorname{Ext}^{\cdot}_{(\mathfrak{g}, K)}(E^*, L'^{\infty}_{P,\omega}) \neq 0$.

(2) I do not know whether β is also surjective. In particular, is $\operatorname{Ext}^i_{(\mathfrak{g}, K)}(E^*, L'^{\infty}_{P,\omega})$ zero outside the interval $(m-(l_0/2), m+(l_0/2)]$?

(3) We already pointed out that the proof of 3.4 in [2] is also valid if the $L_{P,V}$ there is replaced by our $L'_{P,\omega}$. In the same way, 3.5 and 3.6 in [2] and their proofs also hold under that change. In particular, the implication (iv)⇒(ii) of 3.6 shows that $\operatorname{Ext}_{(\mathfrak{g}, K)}(E^*, L'^{\infty}_{P,\omega}) = 0$ if E is not equivalent to $\bar{E}^*$, for any $\omega \in \hat{M}_d$.

In case $E \sim \bar{E}^*$, we want now to show the existence of some $\omega \in \hat{M}_d$ for which $\operatorname{Ext}_{(\mathfrak{g}, K)}(E^*, L'^{\infty}_{P,\omega}) \neq 0$. For this we need to bring a complement to 5.5, 5.7 of [5: II, § 5].

2.9. Remark on [5: II, § 5]. In 5.3, *loc. cit.* it is proved that if M is a connected linear semi-simple group, L a maximal compact subgroup of M, F an irreducible dimensional representation of M, and V a discrete series representation of M, then

$$\operatorname{Ext}^{\cdot}_{(\mathfrak{m}, L)}(F, V) = 0 \quad \text{if} \quad \chi_V \neq \chi_F; \tag{1}$$

$$\dim \operatorname{Ext}^i_{(\mathfrak{m}, L)}(F, V) = \delta_{i,q} \quad (q = (\dim M/L)/2,\ i \in \boldsymbol{Z}) \quad \text{if} \quad \chi_V = \chi_F. \tag{2}$$

It is then shown (5.5, 5.7) that if M is reductive, with compact center, and F, V are as before, then

(3) $$\dim \mathrm{Ext}^i_{(\mathfrak{m}, L)}(F, V) \leqq \delta_{i,q} \quad (i \in \boldsymbol{Z}).$$

We want now to point out that if $M=M_P$ as above, with P cuspidal in G, then, given F, there exists V in the discrete series of M such that

(4) $$\mathrm{Ext}^q_{(\mathfrak{m}, L)}(F, V) = \boldsymbol{C}.$$

Let first M be any connected reductive group with compact center. It is then the almost direct product of a semi-simple group M' by a torus T. The representation F is the tensor product of an irreducible representation F' of M' by a one-dimensional representation $\boldsymbol{C}_\lambda$ of T. Fix a discrete series representation V' of M' with infinitesimal character equal to $\chi_{F'}$. Then, by known results on the L-weights of V', the characters of $T \cap M'$ given by V' and F' are the same. Therefore $V' \otimes \boldsymbol{C}_\lambda$ is also a representation of M, hence an element V of the discrete series of M. Using the Künneth rule, one sees immediately that

(5) $$\mathrm{Ext}^\cdot_{(\mathfrak{m}, L)}(F, V) = \mathrm{Ext}^\cdot_{(\mathfrak{m}', L \cap M')}(F', V'),$$

and we are reduced to (2) above, taking into account the fact that

$$M/L = M'/(M' \cap L).$$

Let now $M=M_P$, with P cuspidal in G. We claim that M is the direct product of M^0 by a finite elementary abelian 2-group, say Z. By our standing assumption G is linear. Let G_c be its complexification. It is an algebraic $\boldsymbol{R}$-group. The group P is of finite index in the group of real points of the parabolic $\boldsymbol{R}$-subgroup $\mathscr{P}$ of G_c with Lie algebra $\mathfrak{p}_c$, and A is the identity component, in ordinary topology, of the group of real points $\mathscr{A}(\boldsymbol{R})$ of the maximal $\boldsymbol{R}$-split torus $\mathscr{A}$ of the radical of $\mathscr{P}$ with Lie algebra $\mathfrak{a}_c$. The group $\mathscr{A}(\boldsymbol{R})$ is the direct product of A by an elementary abelian 2-group Z_0, the group of elements of order $\leqq 2$ of $\mathscr{A}(\boldsymbol{R})$. By a result of Matsumoto (see [4: § 14]), R meets every connected component of $\mathscr{P}(\boldsymbol{R})$. Since $N \cdot A$ is connected, and Z_0 centralizes A, this implies immediately our assertion, with $Z = Z_0 \cap M$.

The representation F is the tensor product of an irreducible representation F^0 of M^0 by a one-dimensional representation $\boldsymbol{C}_\sigma$ of Z. By the previous argument we may find a discrete series representation V^0 of M^0 such that

(6) $$\mathrm{Ext}^\cdot_{(\mathfrak{m}, L^0)}(F^0, V^0) \neq 0$$

where $L^0 = M^0 \cap L$. We then take $V = V^0 \otimes \boldsymbol{C}_\sigma$. Since Z is central, it acts trivially on $\Lambda^\cdot \mathfrak{m}/\mathfrak{l}$, from which it follows that

(7) $$C^\cdot(\mathfrak{m}, L; F \otimes V) = C^\cdot(\mathfrak{m}, L^0; F^0 \otimes V^0),$$

whence

$$\text{(8)} \qquad \operatorname{Ext}^{\cdot}_{(\mathfrak{m}, L)}(F, V) = \operatorname{Ext}^{\cdot}_{(\mathfrak{m}, L^0)}(F^0, V^0),$$

and our assertion.

2.10. Remark. We take this opportunity to correct an oversight in [2]: In the proof of 3.7, we apply [5: II, 5.3] to $\operatorname{Ext}^{\cdot}_{(\mathfrak{m}, K\cap M)}(F^*, V)$ although M is not necessarily connected semi-simple. But $M = M_P$ with P fundamental and G is semi-simple, linear, so that the above holds. Also F^* there stands for $F^*_{s\lambda|\mathfrak{b}_c}$.

3. Proof of Theorems A and B

3.1. By [1: 3.5], there is a canonical inclusion

$$\text{(1)} \qquad C^{\cdot}(\mathfrak{g}, K; L^2(G)^\infty \otimes E) \to A^{\cdot}_{(2)}(K; E),$$

which induces an isomorphism in cohomology. Let us denote by d (resp. d_X) the differential on the left (resp. right)-hand side. The left-hand side is contained in the graded Hilbert space

$$\text{(2)} \qquad C^{\cdot}(\mathfrak{g}, K; L^2(G) \otimes E) = \operatorname{Hom}_K(\Lambda^{\cdot}\, \mathfrak{g}/\mathfrak{k}, L^2(G) \otimes E).$$

Let $\bar{d}$ be the closure of d. Then (1) extends to an isomorphism of the graded Hilbert space $C^{\cdot}(\mathfrak{g}, K; L^2(G) \otimes E)$ onto the space of L^2-forms on X with measurable coefficients, which maps $\operatorname{dom} \bar{d}$ onto $\operatorname{dom} \bar{d}_X$ and $\bar{d}$ onto $\bar{d}_X$ [1: 3.6]. We are therefore reduced to the discussion of $\operatorname{Ext}^{\cdot}_{(\mathfrak{g}, K)}(E^*, L^2(G)^\infty)$ and of the range of $\bar{d}$.

3.2. As in 2.1, let J be the set of K-types occurring in $\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k}) \otimes E$. By 1.4, we can write $L^2(G) = Q \oplus R$, where Q is a sum of finitely many elementary subspaces, R the orthogonal complement to Q and $R_J = 0$. In view of 2.1(2), which is valid for any continuous G-module, we have then

$$\text{(1)} \qquad \operatorname{Ext}^{\cdot}_{(\mathfrak{g}, K)}(E^*, L^\infty_{P,\omega}) = \operatorname{Ext}^{\cdot}_{(\mathfrak{g}, K)}(E^*, R^\infty) = 0 \quad (L_{P,\omega} \subset R),$$

since the complexes which give rise to these cohomology spaces are already zero. We have therefore

$$\text{(2)} \qquad \operatorname{Ext}^{\cdot}_{(\mathfrak{g}, K)}(E^*, L^2(G))^\infty = \oplus_{P \in S, \omega \in \hat{M}_{P,d}} \operatorname{Ext}^{\cdot}_{(\mathfrak{g}, K)}(E^*, L^\infty_{P,\omega}),$$

where the sum on the right-hand has at most finitely many non-zero terms. We can now use the results of Section 2.

3.3. Let first G and K be of equal rank. Then G is its own fundamental parabolic subgroup and has a discrete series. Theorem A now follows from 3.2(2) and 2.3, 2.7.

3.4. Let now $l_0 = \operatorname{rk} G - \operatorname{rk} K$ be $\neq 0$. By 2.7 we may, on the right-hand side, restrict the summation to the unique fundamental parabolic subgroup of G contained in S. Since $L_{P,\omega}$ is unitary, it is standard that $\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L^{\infty}_{P,\omega})=0$ if $E\not\sim\bar{E}^*$, which proves Theorem B in that case. So assume $E\sim\bar{E}^*$. In view of the injectivity of β in 2.8 and of the remark to 2.8, it suffices, to conclude the proof of Theorem B, to show the existence of $\omega\in\hat{M}_{P,d}$ such that $\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L'_{P,\omega})\neq 0$. We use the notation of 2.5. Since P is fundamental, there exists $s\in W^P$ such that 2.5(1) is satisfied [2: 3.6]. Then $\mathrm{Ext}^{\cdot}_{(\mathfrak{g},K)}(E^*, L'^{\infty}_{P,\omega})$ is given by 2.8(6), and it is therefore enough to show the existence of $\omega\in\hat{M}_{P,d}$ such that $\mathrm{Ext}^{\cdot}_{(\mathfrak{m},K\cap M)}(F^*_{s\lambda|\mathfrak{b}_c}, V^{\infty}_{\omega})\neq 0$. But this follows from 2.9.

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