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ON ANALYTIC CONTINUATION FROM THE "EDGE OF THE WEDGE"

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The "edge of the wedge" theorem was first proved by N. N. Bogoliubov in 1956 in connection with applications to quantum field theory (see [7]). During the next years many various proofs, generalizations and refinements of Bogoliubov's theorem were obtained; the problems and results concerning this theorem constituted an important chapter in the theory of analytic functions of several complex variables and its applications (see [1—9]; an extensive bibliography can be found in V. S. Vladimirov's recent survey [9]).

In this note we present some "one-sided" versions of the "edge of the wedge" theorem, which are closely related with Malgrange-Zerner's theorem (see [3]). Our approach is based on the classical ideas and results of R. Nevanlinna and T. Carleman, connected with the notion of the harmonic measure.

1. Notations

Let D_j be a Jordan domain in the complex z_j-plane, E_j an open subset of the boundary bD_j and $F_j=D_j\setminus E_j$ $(j=1, \ldots, n)$. D, E, F (the sets in C^n) are the direct products of D_j, E_j, F_j, $j=1, \ldots, n$, respectively.

$$V_j = E_1\times\ldots\times F_j\times\ldots\times E_n, \quad V_j^\circ = E_1\times\ldots\times D_j\times\ldots\times E_n = V_j\setminus E.$$

$$V=\bigcup_{j=1}^{n} V_j, \quad V^\circ = \bigcup_{j=1}^{n} V_j^\circ = V\setminus E.$$

The harmonic measure of $bD_j\setminus E_j$ with respect to D_j will be denoted by h_j; h_j is the generalized solution of the Dirichlet problem in D_j for the boundary values 0 on E_j and 1 on $bD_j\setminus E_j$. We define h_j on F_j, putting $h_j=0$ on E_j; then h_j is a function, continuous on F_j and harmonic in D_j.

$$h(z) = h(z_1, \ldots, z_n) = \sum_{j=1}^{n} h_j(z_j), \quad z\in F.$$

$W=\{z\in F\colon h(z)<1\}$, $W^\circ=W\setminus E$, $H=\operatorname{int} W$ (the interior of W in C^n).

The "edge" E is an n-dimensional (real) subset of the boundary bW $(=bH)$. The $(n+1)$-dimensional subset V° of bW consists of n parts V_j°; there is a single

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complex direction on each of these parts (the variable z_j on V_j°). All V_j°, $j=1, \dots, n$, are joined on their common border E, the edge of the "wedge" W.

On e, $C(e)$ denotes the class of continuous functions; $CR(e)$, $e=V^\circ$ or W°, denotes the class of continuous CR-functions on e. $|f|_e$ is the sup norm of f on e.

2. Theorems

Theorem 1. *Let* $f\in C(V)\cap CR(V^\circ)$. *Then there exists* $\tilde f\in C(W)\cap CR(W^\circ)$ *such that* $\tilde f=f$ *on* V. *If* $|f|_V<+\infty$, *then the following estimate holds:*

(1) $$|\tilde f(z)| \leqq |f|_E^{1-h(z)}\,|f|_V^{h(z)}, \quad z\in W.$$

The first statement of Theorem 1 means that if $f\in C(E)$ admits CR-continuation from the "edge" E to V, then f admits CR-continuation to the "wedge" W. We note that $\tilde f\in C(W)\cap CR(W^\circ)$ if and only if $\tilde f$ is continuous on W and holomorphic in H. Clearly H is the hull of holomorphy of the set V (W is CR-hull of V); the characterization of the hull of holomorphy of V in terms of the harmonic measures coincide with that of the "cross" in Siciak's theorem on separately analytic functions [6]. The estimate (1) is a version of the "two-constant theorem" for our geometrical configuration.

The harmonic measures h_j are invariant under conformal mappings of the domains D_j; hence Theorem 1 is equivalent to the same assertion for any special choice of the domains D_j, $j=1, \dots, n$. The following realizations are the most interesting:

1° $$D_j = \{z_j\colon \operatorname{Im} z_j > 0\};$$

2° $$D_j = \{z_j\colon 0 < \operatorname{Im} z_j < b_j\}, \quad E_j \subset (-\infty, +\infty);$$

3° $$D_j = \{z_j\colon |\operatorname{Re} z_j| < a_j,\ 0 < \operatorname{Im} z_j < b_j\}, \quad E_j \subset (-a_j, a_j);$$

4° $$D_j = \{z_j\colon |z_j| < 1\}.$$

In these cases, h_j, h and W have a simple geometrical meaning. For instance, in the case 1°, $h_j=1-\alpha_j/\pi$, where $\alpha_j(z_j)$ is the angular measure of the set E_j at the point z_j. If E_j's are (connected) arcs, it is convenient to realize D_j's as in 2° or 3° (putting $E_j=(-\infty, +\infty)$ and $E_j=(-a_j, a_j)$, respectively). In the case 2° with $E_j=(-\infty, +\infty)$ we have: $h_j(z_j)=y_j/b_j$, $h(z)=y_1/b_1+\dots+y_n/b_n$ and W is the convex hull of V. One can obtain more general geometrical versions of Theorem 1 (in the cases 1°—3°) after a linear transformation of the space R_y^n ($C_z^n=R_x^n+iR_y^n$).

Theorem 1 admits various generalizations. The statements about CR-continuation from V° to W° can be proved under weaker assumptions on the existence and coincidence of the boundary values of the functions $f_j=f|_{V_j^\circ}$ on E. Moreover, the behaviour of $\tilde{f}$ for $z\to t^\circ\in E$ depends only on the local properties of the function $f(t)$, $t\in E$, at the point t° (here f is a common boundary value of the functions f_j on E). We will formulate the corresponding theorem for the bounded functions.

Suppose E_j's are rectifiable sets, $f\in CR(V^\circ)$ and $|f|_{V^\circ}<+\infty$; each function $f(\dots, z_j, \dots)$, $z_j\in D_j$ (for any fixed variables $z_k=t_k\in E_k$, $k\neq j$) has nontangential boundary values for almost all $t_j\in E_j$. Hence the functions $f_j=f|_{V_j^\circ}$ define the boundary functions $f_j(t)$, $j=1, \dots, n$, on the "edge" E. Under these assumptions and notations we have the following theorem.

Theorem 2. *Let the boundary functions f_j coincide almost everywhere on E: $f_j(t)=f(t)$ for almost all $t\in E$ with respect to the n-dimensional Lebesque measure. Then there exists $\tilde{f}\in CR(W^\circ)$ such that $\tilde{f}=f$ on V°. If the function $f(t)$, $t\in E$, is continuous at the point $t^\circ\in E$, then $\tilde{f}(z)\to f(t^\circ)$ for $z\to t^\circ$, $z\in W^\circ$.*

The smoothness of $f(t)$, $t\in E$, at the point t° implies almost the same smoothness of $\tilde{f}$ at this point. More precisely, if for some polynomial T_{t° of degree k we have $(f-T_{t^\circ})(t)=O(|t-t^\circ|^{k+s})$, $0<s<1$, for $t\to t^\circ$, $t\in E$, then $(f-T_{t^\circ})(z)=O(|z-t^\circ|^{k+s'})$ for every $s'<s$ and $z\to t^\circ$, $z\in W^\circ$.

3. Method of the proof. Lemmas

Without loss of generality we may assume that E_j's are rectifiable sets and $|f|_E<+\infty$. Let $\tilde{h}_j$ be a conjugate function for the harmonic measure h_j ($\tilde{h}_j(z_j^\circ)=0$ for some fixed point $z_j^\circ\in D_j$), $g_j=h_j+i\tilde{h}_j$ and

$$g(z)=\sum_{j=1}^n g_j(z_j), \quad z\in D.$$

We set

$$K_m(f;\ z)=\frac{1}{(2\pi i)^n}\int_E e^{-m(g(t)-g(z))}\frac{f(t)\,dt}{(t_1-z_1)\dots(t_n-z_n)}, \quad z\in D \tag{2}$$

(each function e^{-g_j} is bounded in D_j; we integrate in (2) nontangential boundary values of the corresponding functions).

If a function f is holomorphic in D and (say) continuous on E, then the following Carleman's formula holds:

$$f(z)=\lim_{m\to\infty}K_m(f;\ z), \quad z\in D; \tag{3}$$

(3) is easy to deduce from Cauchy's formula.

The first (and the main) point of our approach is

Lemma 1. *If* $f \in C(V) \cap CR(V^\circ)$ *(E_j's are rectifiable,* $|f|_E < +\infty$*), then the formula*

$$\tilde{f}(z) = \lim_{m \to \infty} K_m(f;\ z), \quad z \in H = \operatorname{int} W, \tag{4}$$

defines a holomorphic function $\tilde{f}$ *in the domain* H.

The same assertion is true under the condition of Theorem 2. We note the essential difference between the formulas (3) and (4). Carleman's formula (3) *reconstructs a holomorphic function* f in the domain D by its boundary values on the part E of bD. The formula (4) *defines holomorphic continuation* of the function $f(t)$, $t \in E$, in the domain H if it is known that the function admits CR-continuation on V° — on the parts V_j°, $j=1, \ldots, n$, of the boundary bH with minimal complex structure (a single complex direction — variable z_j — on each of these parts).

Using the formulas of the type (4) one can define $\tilde{f}$ on all parts of bH with at least two complex directions. On V_j°'s the corresponding formulas reconstruct f (for $f \in C(\mathscr{C}) \cap CR(V^\circ)$, Carleman's one-dimensional formula works on these parts of bH). We put $\tilde{f}=f$ on V.

A comparison of these formulas provides the proof of the continuity of $\tilde{f}$ on W°; using this fact one can prove the equation $|\tilde{f}|_{W^\circ}=|f|_{V^\circ}$. Finally, the continuity of $\tilde{f}(z)$, $z \in W$, at the points $t \in E$ can be proved with the help of the following analogue of Nevanlinna's "two-constant theorem".

Lemma 2. *Let* $f \in CR(W^\circ)$, $|f|_{W^\circ}=M<+\infty$, *and for some fixed* $j \in \{1, \ldots, n\}$ *and any* $t \in E$

$$\limsup_{z \to t} |f_j(z)| \leqq m, \quad f_j = f | V_j^0.$$

Then the following estimate holds:

$$|f(z)| \leqq m^{1-h(z)} M^{h(z)}, \quad z \in W^0.$$

References

[1] BEURLING, A.: Analytic continuation across a linear boundary. - Acta Math. 128, 1972, 153—182.

[2] EPSTEIN, H.: Generalization of the "edge of the wedge" theorem. - J. Math. Phys. 1, 1960, 524—531.

[3] EPSTEIN, H.: Some analytic properties of scattering amplitudes in quantum field theory. - Particle symmetries and axiomatic field theory. Vol. 1: Axiomatic field theory. Brandeis University Summer Institute in Theoretical Physics, 1965. Gordon and Breach, New York—London—Paris, 1966, 1—133.

[4] MARTINEAU, A.: Le "edge of the wedge theorem" en théorie des hyperfonctions de Sato. - Proc. International Conference on Functional Analysis and Related Topics. Tokyo, 1969. Univ. of Tokyo Press, Tokyo, 1970, 95—106.

[5] Rudin, W.: Lectures on the edge-of-the-wedge theorem. - Regional Conference Series in Mathematics 6. American Mathematical Society, Providence, R. I., 1971.

[6] Siciak, J.: Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of C^n. - Ann. Polon. Math. 22, 1969/70, 145—171.

[7] Vladimirov, V. S.: Methods in the theory of functions of several complex variables. - Izdat. "Nauka", Moscow, 1964 (Russian).

[8] Vladimirov, V. S.: Bogoliubov's "edge of the wedge" theorem, its development and applications. - Problems of theoretical physics. Izdat. "Nauka", Moscow, 1969, 61—67 (Russian).

[9] Vladimirov, V. S.: Functions of several complex variables in mathematical physics. - Problems of mathematics and mechanics. Izdat. "Nauka", Novosibirsk, 1983, 15—32 (Russian).

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