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# ON THE SPHERICAL DERIVATIVE OF SMOOTHLY GROWING MEROMORPHIC FUNCTIONS WITH A NEVANLINNA DEFICIENT VALUE

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## 1. Introduction and results

Let $f$ be meromorphic in the finite complex plane $C$. We write

$$\varrho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2}$$

and

$$\mu(r,f) = \sup\{\varrho(f(z)):\ |z| = r\}.$$

We shall use the usual notations of the Nevanlinna theory.

Clunie and Hayman [3] proved the following result.

Theorem A. *If $\varphi(r)$ is positive and increasing and $f(z)$ is a transcendental entire fuction such that*

$$\log M(r,f) = O\left(\frac{(\log r)^2}{\varphi(r)}\right) \quad (r\to\infty),$$

*then*

$$\limsup_{r\to\infty} \frac{r\mu(r,f)}{\varphi(r)\log r} = \infty.$$

This result was extended in [9] for functions which have a Nevanlinna deficient value in the following form.

Theorem B. *Let $1<t<2$ and let $f$ be a transcendental meromorphic function such that $\delta(\infty,f)>0$ and that*

(1.1) $$T(r,f) = O((\log r)^t) \quad (r\to\infty).$$

*Then*

(1.2) $$\limsup_{r\to\infty} \frac{\log(r\mu(r,f))}{(\log r)^{2-t}} = \infty.$$

If $f$ satisfies

(1.3) $$T(r,f) = O((\log r)^2) \quad (r\to\infty),$$

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then it satisfies

$$\lim_{r\to\infty} \frac{T(2r,f)}{T(r,f)} = 1, \tag{1.4}$$

too. We shall prove the following extension for Theorem B.

Theorem 1. *Let $f$ be a transcendental meromorphic function satisfying* (1.4) *such that $\delta(\infty,f)>0$. Then*

$$\limsup_{r\to\infty} \frac{r\mu(r,f)}{T(r,f)} = \infty \tag{1.5}$$

*and*

$$\limsup_{r\to\infty} \frac{n(r,0,f)\log(r\mu(r,f))}{T(r,f)} \geqq \delta(\infty,f). \tag{1.6}$$

*If, further, $\delta(\infty,f)>0$ and $f$ satisfies* (1.1) *for some $t$, $1<t\leqq 2$, then*

$$\limsup_{r\to\infty} \frac{\log(r\mu(r,f))}{T(r,f)(\log r)^{1-t}} > 0. \tag{1.7}$$

Clunie and Hayman [3] proved that, given an increasing function $\varphi(r)$ such that $\varphi(r)\to\infty$ as $r\to\infty$, there is a transcendental entire function $f$ satisfying (1.3) such that

$$r\mu(r,f) = O(\varphi(r)\log r) \quad (r\to\infty).$$

This example shows, since

$$\log r = o(T(r,f)) \quad (r\to\infty),$$

that (1.5) is essentially sharp. The following theorem shows that (1.2), (1.6) and (1.7) are essentially sharp.

Theorem 2. *Let $t$, $1<t<2$, and $d$, $0<d\leqq 1$, be given, and let $\varphi(r)$ be an increasing function of $r$ such that $\varphi(r)\to\infty$ as $r\to\infty$. There exists a transcendental meromorphic function $f$ satisfying* (1.1) *such that $\delta(\infty,f)=d$,*

$$\log(r\mu(r,f)) = O(\varphi(r)(\log r)^{2-t}) \quad (r\to\infty), \tag{1.8}$$

$$\limsup_{r\to\infty} \frac{n(r,0,f)\log(r\mu(r,f))}{T(r,f)} = \delta(\infty,f) \tag{1.9}$$

*and*

$$\limsup_{r\to\infty} \frac{\log(r\mu(r,f))}{T(r,f)(\log r)^{1-t}} < \infty. \tag{1.10}$$

The following result shows that (1.3) is the best possible growth condition under which (1.5) holds.

Theorem 3. *Let $\varphi(r)$ and $d$ be as in Theorem 2. There exists a transcendental meromorphic function $f$ such that $\delta(\infty,f)=d$,*

$$T(r,f) = O(\varphi(r)(\log r)^2) \quad (r\to\infty) \tag{1.11}$$

*and*

$$\limsup_{r\to\infty}\frac{r\mu(r,f)}{T(r,f)} \leqq 5\delta(\infty,f). \tag{1.12}$$

It is proved in [14] that if $f$ is a transcendental meromorphic function of order $\lambda$, then

$$\limsup_{r\to\infty}\frac{r\mu(r,f)}{T(r,f)} \geqq A_0(1+\lambda)\delta(\infty,f), \tag{1.13}$$

where $A_0>0$ is an absolute constant, and in [11] a counter example is given which shows that (1.13) is essentially sharp for $0<\lambda<\infty$ and $0<\delta(\infty,f)\leqq 1$. Our Theorem 3 shows that (1.13) is essentially sharp for $\lambda=0$ and $0<\delta(\infty,f)\leqq 1$. The question "which is the best possible value for $A_0$ in (1.13)" remains open.

## 2. Proof of Theorem 1

Let $f$ be as in Theorem 1. We write

$$L(r,f) = \min\{|f(z)|:\ |z| = r\}.$$

It follows from Lemma 1 of [7] that

$$n(r,a,f) = o\big(T(r,f)\big)\quad (r\to\infty) \tag{2.1}$$

for all complex values $a$ and that there exist sequences $x_k$ and $r_k$ such that $1<x_k<r_k<2x_k<x_{k+1}$, $L(x_k,f)=0$ and $L(r_k,f)\geqq 2$ for any $k$,

$$\lim_{k\to\infty} r_k/x_k = 1, \tag{2.2}$$

and that

$$\log L(r_k,f) \geqq \big(\delta(\infty,f)+o(1)\big)T(r_k,f)\quad (k\to\infty). \tag{2.3}$$

For any $k$, we choose $z_k$ such that $x_k<|z_k|<r_k$, $|f(z_k)|=1$, and that $|f(z)|>1$ for $|z_k|<|z|\leqq r_k$. If $|z|=|z_k|$ or $|z|=r_k$, then

$$\log|f(z)| \geqq \frac{\log|z/z_k|}{\log|r_k/z_k|}\log L(r_k,f), \tag{2.4}$$

and since $\log|f(z)|$ is superharmonic on $|z_k|\leqq|z|\leqq r_k$, we deduce that (2.4) holds for all $z$ lying in $|z_k|<|z|<r_k$.

Let $s>0$. From (2.4) it follows that

$$\log|f(z_k(1+s/|z_k|))| \geqq \frac{\log L(r_k,f)}{\log(r_k/x_k)}(s/|z_k|+o(s))$$

as $s\to 0$, and since

$$\log|f(z_k+s(z_k/|z_k|))|$$
$$\leqq \log(|f(z_k)|+(s+o(s))|f'(z_k)|) \leqq (s+o(s))|f'(z_k)|\quad (s\to 0),$$

we deduce that

$$|z_k|\varrho(f(z_k)) = |z_k/2||f'(z_k)| \geqq (2\log(r_k/x_k))^{-1}\log L(r_k, f).$$

This together with (2.2) and (2.3) shows that

$$\frac{|z_k|\,\varrho(f(z_k))}{T(|z_k|, f)} \to \infty \quad \text{as} \quad k \to \infty, \tag{2.5}$$

which proves (1.5).

Let $a_s$ be the zeros of $f$. For any $k$, we choose $w_k$ such that $f(w_k)=0$, $x_k \leqq |w_k| < r_k$, and that $f(z)\neq 0$ for $|w_k|<|z|\leqq r_k$. Since $L(r_k, f)>1$, there exists $d_k$, $0<d_k< r_k/|w_k|-1$, such that $|f(w_k(1+d_k))|=1$ and that

$$|f(w_k(1+d))| < 1 \quad \text{for} \quad 0 < d < d_k. \tag{2.6}$$

Applying the Poisson–Jensen formula with $R=r_k$ and $w=w_k(1+d_k)=re^{i\varphi}$, we get

$$\begin{aligned} 0 &= \log|f(w)| \\ &\geqq (2\pi)^{-1}\int_0^{2\pi}\log|f(\mathrm{R}e^{i\alpha})|\frac{R^2-r^2}{R^2+r^2-2rR\cos(\varphi-\alpha)}\,d\alpha \\ &\qquad -\sum\nolimits_{|a_s|<R}\log\left|\frac{R^2-\bar{a}_s w}{R(w-a_s)}\right| \\ &\geqq \log L(r_k, f) - n(|w_k|, 0, f)\log\frac{2r_k}{|w-w_k|}, \end{aligned}$$

which together with (2.3) implies that

$$n(|w_k|, 0, f)\log(4/d_k) \geqq (\delta(\infty, f)+o(1))T(r_k, f)$$

as $k\to\infty$. This implies that

$$\log(4/d_k) \geqq (\delta(\infty, f)+o(1))\frac{T(r_k, f)}{n(|w_k|, 0, f)} \tag{2.7}$$

as $k\to\infty$.

Since $|f(w)|=1$ and $f(w_k)=0$, there exists $b_k=w_k(1+d)$ such that $0<d<d_k$ and that

$$|f'(b_k)| \geqq |w-w_k|^{-1} = |d_k w_k|^{-1}.$$

This together with (2.7) and (2.1) implies that

$$\begin{aligned} \log(|b_k|\varrho(f(b_k))) &\geqq \log(|b_k/2||f'(b_k)|) \geqq \log(2d_k)^{-1} \\ &\geqq O(1)+(\delta(\infty, f)+o(1))\frac{T(r_k, f)}{n(|w_k|, 0, f)} \\ &\geqq (\delta(\infty, f)+o(1))\frac{T(|b_k|, f)}{n(|b_k|, 0, f)} \quad (k\to\infty), \end{aligned}$$

which proves (1.6).

Let us suppose that $f$ satisfies (1.1) for some $t$, $1<t\leqq 2$, and let $r>1$. We get

$$n(r, 0, f) \leqq (\log r)^{-1} \int_r^{r^2} n(t, 0, f) t^{-1} dt$$
$$= O((\log r)^{-1} N(r^2, 0, f)) = O((\log r)^{t-1}) \quad (r \to \infty),$$

which together with (1.6) proves (1.7). This completes the proof of Theorem 1.

Remark. From (2.5) we get the following result slightly stronger than (1.5).

Theorem 4. *Let $f$ be as in Theorem* 1. *Then*

$$\limsup_{\substack{z \to \infty \\ z \in E(f)}} \frac{|z| \varrho(f(z))}{T(|z|, f)} = \infty,$$

*where* $E(f)=\{z\colon |f(z)|=1\}$.

## 3. Proof of Theorem 2

Let $t$, $d$ and $\varphi(r)$ be as in Theorem 2. We set $r_0=100$, and for $n\geqq 1$ we choose $r_n$ such that

(3.1) $$r_n > \exp\exp\exp(r_{n-1})$$

and

(3.2) $$\varphi(r_n/2) > r_{n-1}.$$

We denote by $[x]$ the integral part of a non-negative real number $x$. We set

(3.3) $$s_n = [(\log r_n)^{t-1}],$$

(3.4) $$q_n = [s_n(1-d)]$$

and

$$f(z) = \Pi_{n=1}^{\infty} \frac{(1-z/r_n)^{s_n}}{(1+z/r_n)^{q_n}}.$$

If $d=1$, then $f$ is an entire function.

Let $r_n^{1/2} \leqq |z| \leqq r_{n+1}^{1/2}$. We have

(3.5) $$\log|f(z)| = (1+o(1))(s_{n-1}-q_{n-1}) \log|z| + s_n \log|(z-r_n)/r_n| + q_n \log|r_n/(z+r_n)|.$$

Let $0<\varepsilon<1/9$. We set

$$D_n = r_n \exp((-1+\varepsilon) s_n^{-1}(s_{n-1}-q_{n-1}) \log r_n)$$

and

$$d_n = r_n \exp((-1-\varepsilon) s_n^{-1}(s_{n-1}-q_{n-1}) \log r_n).$$

It follows from (3.5) that

$$\log|z^2 f(z)| \leqq (-\varepsilon+o(1))(s_{n-1}-q_{n-1})\log|z| \tag{3.6}$$

in $|z-r_n|<d_n$ and that

$$\log|z^{-2}f(z)| \geqq (\varepsilon+o(1))\log|z| \tag{3.7}$$

as $z\to\infty$ outside the union of the discs $|z-r_n|<D_n$.

From (3.7) we deduce that

$$\varrho(f(z)) \leqq \frac{|f'(z)|}{|f(z)|^2} = \left|(2\pi i)^{-1}\int_{|w-z|=1}\frac{1/f(w)}{(w-z)^2}\,dw\right| \leqq (1+o(1))|z|^{-2} \tag{3.8}$$

as $z\to\infty$ outside the union of the discs $|z-r_n|<1+D_n$, and, similarly, from (3.6) we get

$$\varrho(f(z)) \leqq |f'(z)| \leqq (1+o(1))|z|^{-2} \quad (n\to\infty) \tag{3.9}$$

in $|z-r_n|<d_n-1$.

Let $d_n-1\leqq|z-r_n|\leqq 1+D_n$. We have

$$\varrho(f(z)) \leqq |f'(z)/f(z)| = \left|\sum_{p=1}^{\infty}\left(\frac{s_p}{z-r_p}-\frac{q_p}{z+r_p}\right)\right|$$

$$\leqq \frac{(1+o(1))s_n}{d_n-1} = (1+o(1))d_n^{-1}s_n \quad (n\to\infty),$$

which implies together with (3.3) and (3.4) that

$$\log(|z|\varrho(f(z))) \leqq \log(s_n r_n d_n^{-1})+o(1) \tag{3.10}$$

$$= O(\log\log r_n)+(1+\varepsilon)\frac{s_{n-1}-q_{n-1}}{s_n}\log r_n$$

$$\leqq (1+\varepsilon+o(1))s_n^{-1}d\,s_{n-1}\log r_n \quad (n\to\infty).$$

Combining the estimates (3.8), (3.9) and (3.10), we deduce that

$$\log(r\mu(r,f)) \leqq -1+o(1) \tag{3.11}$$

as $r\to\infty$ outside the union of the intervals $r_k/2<r<2r_k$ and that

$$\log(r\mu(r,f)) \leqq (1+\varepsilon+o(1))\frac{d\,s_{k-1}\log r_k}{s_k} \quad (k\to\infty) \tag{3.12}$$

for $r_k/2<r<2r_k$. Since we get (3.12) for all $\varepsilon>0$ and

$$n(2r_k,0,f) = (1+o(1))s_k \quad (k\to\infty),$$

we deduce that

$$\log(r\mu(r,f)) \leqq (d+o(1))\frac{s_{k-1}\log r_k}{n(2r_k,0,f)} \quad (k\to\infty) \tag{3.13}$$

for $r_k/2<r<2r_k$.

It follows from the first main theorem of the Nevanlinna theory and (3.7) that

$$T(2r_k, f) = (1+o(1))N(2r_k, 0, f) = (1+o(1))s_{k-1} \log r_k$$
$$= (1+o(1))N(r_k/2, 0, f) = (1+o(1))T(r_k/2, f) \quad (k \to \infty). \tag{3.14}$$

From (3.4) we deduce that

$$N(r, \infty, f) = (1-d+o(1))N(r, 0, f) \quad (r \to \infty),$$

which together with (3.14) implies that $\delta(\infty, f) = d$. This together with (3.13), (3.14) and (3.11) shows that

$$\log (r\mu(r, f)) \leqq (\delta(\infty, f)+o(1)) \frac{T(r, f)}{n(r, 0, f)} \tag{3.15}$$

as $r \to \infty$. From (3.11), (3.12), (3.14) and (3.3) we deduce that

$$\log (r\mu(r, f)) = O\left(\frac{T(r, f)}{(\log r)^{t-1}}\right) \quad (r \to \infty),$$

which proves (1.10), and from (3.11), (3.12), (3.2) and (3.3) we get

$$\log (r\mu(r, f)) = O(\varphi(r)(\log r)^{2-t}) \quad (r \to \infty),$$

which proves (1.8).

From (3.3) we deduce that

$$n(r, 0, f) = O((\log r)^{t-1}) \quad (r \to \infty),$$

which together with (3.7) and (3.14) implies that

$$T(r, f) = (1+o(1))N(r, 0, f)$$
$$= O\left(\int_1^r (\log x)^{t-1} x^{-1}\, dx\right)$$
$$= O((\log r)^t) \quad (r \to \infty).$$

Combining (3.15) and (1.6) we get (1.9). Theorem 2 is proved.

## 4. Some lemmas

Lemma 1. *Let* $\varphi(r)$ *be an increasing function of* $r$ *such that* $\varphi(r) \to \infty$ *as* $r \to \infty$. *We choose* $r_0 = s_0 = 100$, *and for* $n \geqq 1$, $r_n$ *and* $s_n$ *are chosen such that*

$$r_n > \exp\exp\exp(r_{n-1}), \tag{4.1}$$

$$\varphi(\sqrt{r_n}) > s_{n-1} \tag{4.2}$$

*and*

$$s_n = [s_{n-1} \log r_n]. \tag{4.3}$$

*We set*

$$g(z) = \prod_{n=1}^{\infty} (1-z/r_n)^{s_n}.$$

*Then*

$$T(r, g) = O(\varphi(r)(\log r)^2) \quad (r \to \infty), \tag{4.4}$$

$$\limsup_{r\to\infty} \frac{r\mu(r, g)}{N(r/9, 0, g)} \leqq 5, \tag{4.5}$$

$$N(r_p/9, 0, g) = (1+o(1))s_p \quad (p \to \infty), \tag{4.6}$$

*and*

$$z^2 g(z) \to 0 \tag{4.7}$$

*as* $z \to \infty$ *through the union of the discs* $|z-r_p| \leqq r_p/3$.

*Proof.* Let $r_p/4 \leqq |z| \leqq 4r_p$. We have

$$\log |g(z)| = (1+o(1))s_{p-1} \log r_p + s_p \log |(z-r_p)/r_p| \quad (p \to \infty). \tag{4.8}$$

If $|z-r_p| \geqq 2r_p/5$, we get from (4.3) and (4.8)

$$\begin{aligned} \log |z^{-3} g(z)| &\geqq (1+o(1))s_{p-1} \log r_p - s_p \log (5/2) \\ &= (1-\log(5/2)+o(1))s_{p-1} \log r_p \geqq 1+o(1) \quad (p \to \infty). \end{aligned} \tag{4.9}$$

If $|z-r_p| \leqq r_p/3$, we deduce from (4.3) and (4.8) that

$$\begin{aligned} \log |z^3 g(z)| &\leqq (1+o(1))s_{p-1} \log r_p - s_p \log 3 \\ &= -(\log 3 - 1 + o(1))s_{p-1} \log r_p \leqq -1+o(1) \quad (p \to \infty), \end{aligned} \tag{4.10}$$

which proves (4.7).

Using the minimum principle, we deduce from (4.9) that

$$|z^{-2} g(z)| \to \infty \tag{4.11}$$

as $z \to \infty$ outside the union of the discs $|z-r_p| < 2r_p/5$.

From (4.11) it follows that

$$\begin{aligned} |z|\varrho(g(z)) &\leqq |zg'(z)||g(z)|^{-2} \\ &= \left| (2\pi i)^{-1} z \int_{|w-z|=1} \frac{1/g(z)}{(w-z)^2} \, dw \right| = o(1) \end{aligned} \tag{4.12}$$

as $z \to \infty$ outside the union of the discs $|z-r_p| < r_p/2$, and from (4.7) we get

$$\begin{aligned} |z|\varrho(g(z)) &\leqq |zg'(z)| \\ &= \left| (2\pi i)^{-1} z \int_{|w-z|=1} g(z)(w-z)^{-2} \, dw \right| = o(1) \end{aligned} \tag{4.13}$$

as $z \to \infty$ through the union of the discs $|z-r_p| \leqq r_p/4$.

Let $r_p/4<|z-r_p|<r_p/2$. We have

$$|z|\varrho(g(z)) \leqq |zg'(z)/g(z)| = |z\textstyle\sum_{k=1}^{\infty} s_k(z-r_k)^{-1}| \tag{4.14}$$

$$\leqq s_p|z|\,|z-r_p|^{-1}+4s_{p-1}+o(1) \leqq (5+o(1))s_p \quad (p\to\infty).$$

Let $r_p/100<r\leqq r_p$. From (4.1) and (4.3) we get

$$N(r,0,f) = (1+o(1))s_{p-1}\log r \tag{4.15}$$

$$= (1+o(1))s_{p-1}\log r_p = (1+o(1))s_p \quad (p\to\infty),$$

which proves (4.6), and together with (4.14), (4.12) and (4.13) shows that

$$r\mu(r,f) \leqq (5+o(1))N(r/9,0,f) \quad (r\to\infty),$$

which proves (4.5).

From (4.1), (4.2) and (4.3) we get for $r_p^{1/2}\leqq r\leqq r_{p+1}^{1/2}$

$$T(r,f) \leqq (1+o(1))\log M(r,f) \leqq (1+o(1))s_p\log r$$

$$\leqq (2+o(1))s_{p-1}(\log r)^2 \leqq (2+o(1))\varphi(r)(\log r)^2,$$

which proves (4.4). Lemma 1 is proved.

The following lemma is proved in [11].

Lemma 2. *Let $k$ be a positive integer, $g(z)=(1-z^{8k})^{-1}$, $g_p(z)=g(2^{-p/k}z)$ for $p=1,\dots,k$, and*

$$f_k(z) = \textstyle\sum_{p=1}^{k}(-1)^p g_p(z).$$

*Then $n(r,\infty,f_k)=8k^2$ for $r\geqq 2$,*

$$\varrho(f_k(z)) < 72k \tag{4.16}$$

*for all $z$ in the finite complex plane $C$, and if $|z|\geqq 4$, then*

$$|f_k(z)| \leqq |2/z|^{6k}. \tag{4.17}$$

## 5. Proof of Theorem 3

Let $\varphi(r)$ and $d$ be as in Theorem 3. If $d=1$, we choose $f(z)=g(z)$, where $g$ is the function of Lemma 1, and deduce from Lemma 1 that $f$ satisfies the assertions of Theorem 3.

Let us suppose that $0<d<1$. Let $g$, $s_p$ and $r_p$ be as in Lemma 1. We set

$$b = 1/d-1, \tag{5.1}$$

$$q_p = 1+[(bs_p/8)^{1/2}], \tag{5.2}$$

$$h_p(z) = f_{q_p}(8r_p^{-1}p^2(z-r_p)),$$

where $f_{q_p}$ is as in Lemma 2, and

$$h(z) = \sum_{p=1}^{\infty} h_p(z).$$

We set $f=g+h$.

It follows from Lemma 2 that

$$\varrho(h_p(z)) = 8r_p^{-1}p^2\varrho(f_{q_p}(8r_p^{-1}p^2(z-r_p))) \leqq 576r_p^{-1}p^2q_p \tag{5.3}$$

for all $z$, and if $|z-r_p| \geqq r_p/p$, then

$$|h_p(z)| \leqq \left|\frac{r_p}{4p^2(z-r_p)}\right|^{6q_p} \leqq \min(p^{-2}, r_p|z-r_p|^{-1}). \tag{5.4}$$

Since the series $\sum p^{-2}$ is convergent and, for any fixed $p$, $r_p|z-r_p|^{-1} \to 0$ as $z \to \infty$, we deduce from (5.4) that

$$|h(z)| \leqq \sum_{p=1}^{\infty} |h_p(z)| \to 0 \tag{5.5}$$

as $z \to \infty$ outside the discs $|z-r_p| < r_p/p$, and that

$$|h(z)-h_p(z)| \leqq o(1) \quad (p \to \infty) \tag{5.6}$$

in $|z-r_p| \leqq r_p/2$.

Let $|z-r_p| \leqq r_p/3$. We write $f(z)=h_p(z)+H_p(z)$. Since $H_p=g+h-h_p$, we deduce from (5.6) and Lemma 1 that

$$|H_p(z)| \leqq o(1) \quad (p \to \infty) \tag{5.7}$$

and, integrating along the circle $|w-r_p|=r_p/3$, that

$$|H_p'(z)| = \left|(2\pi i)^{-1} \int H_p(w)(w-z)^{-2}\, dw\right| \leqq o(r_p^{-1}) \quad (p \to \infty) \tag{5.8}$$

in $|z-r_p| \leqq r_p/6$. Since

$$\varrho(f(z)) \leqq \frac{|h_p'(z)|}{1+|h_p(z)+H_p(z)|^2} + |H_p'(z)|,$$

we get from (5.7) and (5.8)

$$\varrho(f(z)) \leqq (1+o(1))\varrho(h_p(z)) + o(r_p^{-1}) \quad (p \to \infty),$$

which together with (5.3), (5.2) and Lemma 1 implies that

$$|z|\varrho(f(z)) \leqq O(p^2 q_p) = O(p^2 s_p^{1/2}) = o(s_p) = o(N(|z|, 0, g)) \quad (p \to \infty) \tag{5.9}$$

in $|z-r_p| \leqq r_p/6$.

Integrating along the circle $|w-z|=|z|/24$, we deduce from (5.5) that

$$|h'(z)| = \left|(2\pi i)^{-1} \int h(z)(w-z)^{-2}\, dw\right| = o(|z|^{-1}) \tag{5.10}$$

as $z \to \infty$ outside the discs $|z-r_p| < r_p/6$. Since

$$\varrho(f(z)) \leqq \frac{|g'(z)|}{1+|g(z)+h(z)|^2} + |h'(z)|,$$

we get from (5.5), (5.10) and Lemma 1

$$|z|\varrho(f(z)) \leqq (1+o(1))|z|\,\varrho(g(z))+o(1) \leqq (5+o(1))N(|z|, 0, g) \tag{5.11}$$

as $z\to\infty$ outside the discs $|z-r_p|<r_p/6$. Combining (5.9) and (5.11) we get

$$r\mu(r,f) \leqq (5+o(1))N(r, 0, g) \quad (r\to\infty). \tag{5.12}$$

Let $r_p(1-1/p)\leqq r\leqq r_{p+1}(1-(p+1)^{-1})$. It follows from (5.2) and Lemma 2 that

$$\begin{aligned} N(r, \infty, h) &\leqq (8+o(1))\,q_{p-1}^2\log r+8q_p^2\log\left(r/(r_p-r_p/p)\right) \\ &= (b+o(1))s_{p-1}\log r+(b+o(1))s_p\log^+(r/r_p) \quad (p\to\infty) \end{aligned} \tag{5.13}$$

and that

$$\begin{aligned} N(r, \infty, h) &\geqq (8+o(1))\,q_{p-1}^2\log r+8q_p^2\log^+\left(r/(r_p+r_p/p)\right) \\ &= (b+o(1))s_{p-1}\log r+(b+o(1))s_p\log^+(r/r_p) \quad (p\to\infty). \end{aligned} \tag{5.14}$$

Since

$$N(r, 0, g) = (1+o(1))s_{p-1}\log r+s_p\log^+(r/r_p),$$

for these values of $r$ we deduce that

$$\begin{aligned} N(r, \infty, h) &= (b+o(1))N(r, 0, g) \\ &= (b+o(1))s_{p-1}\log r+(b+o(1))s_p\log(r/r_p) \quad (p\to\infty) \end{aligned} \tag{5.15}$$

for $r_p\leqq r\leqq r_{p+1}$.

Using the first main theorem of the Nevanlinna theory, we deduce from (5.5), (5.13) and (5.14) that if $r_p(1-1/p)\leqq r\leqq r_p(1+1/p)$ then

$$\begin{aligned} m(r, \infty, h) &= T(r, h)-N(r, \infty, h) \\ &\leqq T(r_p(1+1/p), h)-N(r_p(1-1/p), \infty, h) \\ &= N(r_p(1+1/p), h)-N(r_p(1-1/p), h)+o(1) \\ &= o(s_{p-1}\log r)+o(s_p) \\ &= o(N(r, 0, g)) \quad (p\to\infty), \end{aligned}$$

which together with (5.5) implies that

$$m(r, h) = o(N(r, 0, g)) = o(T(r,f)) \quad (r\to\infty). \tag{5.16}$$

Since $g$ is an entire function and

$$|m(r,f)-m(r, g)| \leqq m(r, h)+\log 2,$$

we get from (5.16)

$$\begin{aligned} m(r,f) &= m(r, g)+o(N(r, 0, g)) \\ &= (1+o(1))T(r, g) \quad (r\to\infty). \end{aligned} \tag{5.17}$$

Since $N(r,f)=N(r,h)$ for all $r>0$, we get from (5.15) and (5.17)

$$\frac{m(r,f)}{T(r,f)} = \frac{T(r,g)}{T(r,g)+bN(r,0,g)}+o(1)$$
$$\geqq (1+b)^{-1}+o(1) = d+o(1) \quad (r\to\infty),$$

which implies that $\delta(\infty,f)\geqq d$, and since

$$T(r_p^2,g) = \big(1+o(1)\big)N(r_p^2,0,g) \quad (p\to\infty),$$

we get

$$\frac{m(r_p^2,f)}{T(r_p^2,f)}\to d \quad \text{as} \quad p\to\infty,$$

which implies that $\delta(\infty,f)\leqq d$. These estimates imply that $\delta(\infty,f)=d$.

Since $\delta(\infty,f)>0$, it follows from (5.17) and Lemma 1 that

$$T(r,f) = O\big(m(r,f)\big) = O\big(T(r,g)\big)$$
$$= O\big(\varphi(r)(\log r)^2\big) \quad (r\to\infty),$$

which proves (1.11).

From (5.12), (5.15) and (5.17) we get

$$\frac{r\mu(r,f)}{T(r,f)} \leqq \frac{5N(r,0,g)}{bN(r,0,g)+T(r,g)}+o(1)$$
$$\leqq 5(1+b)^{-1}+o(1) = 5d+o(1) \quad (r\to\infty),$$

which proves (1.12). Theorem 3 is proved.

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