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A UNIQUENESS THEOREM FOR THREE MEROMORPHIC FUNCTIONS

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Dedicated to Professor Lars Ahlfors
on the occasion of his 80^{th} birthday

1. Introduction

We say that two nonconstant meromorphic functions f and g share the value a if the zeros of $f-a$ and $g-a$ ($1/f$ and $1/g$ if $a=\infty$) are the same. The following uniqueness theorems are already classical:

Theorem A (R. Nevanlinna [6], p. 372). *If two meromorphic functions in the plane share five distinct values, then both functions agree.*

The number five is best possible as is shown by $f=e^z$ and $g=e^{-z}$ which share the values -1, 0, 1 and ∞.

Theorem B (R. Nevanlinna [6], p. 378). *If two distinct nonconstant meromorphic functions f and g share four values by counting multiplicities, then two of the values, a_1 and a_2, say, are Picard values, and*

$$(f,g,a_3,a_4)=(a_1,a_2,a_3,a_4)=-1$$

holds.

Here, as usual, $(a,b,c,d)=(a-c)/(b-c):(a-d)/(b-d)$ denotes the cross-ratio.

For refinements and generalizations of Theorem B the reader is referred to G. Gundersen [2] and E. Mues (see the book of G. Jank and L. Volkmann [5]).

In [1] H. Cartan considered the problem whether three functions may share four values, and he came to the conclusion that this is possible only in the trivial case that two of the functions are equal.

However, the proof in [1] contains a serious gap, and the first purpose of this note was to fill out this gap. Short before having reached this goal, it turned out that the theorem of Cartan is false. Instead of Cartan's theorem the following is true:

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Theorem 1. *Let f, g and h be dinstinct nonconstant meromorphic functions in the plane sharing the values $1, 0, \infty$ and $-a$. Then $a \neq 1$ is a third root of unity and $w = f, g$ and h are solutions of the algebraic equation*

$$P(U(z), w) = 0, \tag{1}$$

where

$$P(x,y) = y^3 - 3\big((\bar{a}-1)x^2 - 2x\big)y^2 - 3\big(2x^2 - (a-1)x\big)y - x^3, \tag{2}$$

and the meromorphic function U is a nonconstant solution of the differential equation

$$\left(\frac{dU}{dz}\right)^2 = 4\gamma'^2 U(U+1)(U-a), \tag{3}$$

where γ is a nonconstant entire function.

We remark that the general case, where f, g and h share the values a_1, a_2, a_3, a_4 can be reduced to Theorem 1 by the Möbius transform $w \mapsto (w, a_2, a_3, a_4)$.

The converse of Theorem 1 is also true.

Theorem 2. *Let $a \neq 1$ be a third root of unity and let γ be a nonconstant entire function. Then every solution of (3) is meromorphic in the plane, and if U is an arbitrary nonconstant solution, then there exist three distinct meromorphic functions in the plane which satisfy equation (1) and share the values $1, 0, \infty$ and $-a$.*

The simplest case is $\gamma(z) = z$, and if f, g and h form a solution of our problem, then the most general solution is $f \circ \gamma$, $g \circ \gamma$ and $h \circ \gamma$, where γ is an arbitrary nonconstant entire function.

Let U be a solution of $(dU/dz)^2 = 4U(U+1)(U-a)$ having a (double) pole at $z = 0$. Then the poles of U form the period module of U, and if ω_1 and ω_2 are poles of f of order four, then clearly $\omega_2 - \omega_1$ is a period of f (and so of g and h). Since asymptotically every third pole is a pole of f of order four, f (and so g and h) are doubly-periodic with elliptic order six, while the elliptic order of U is two. Thus the area of a parallelogram of primitive periods of f is three times the area of the corresponding period parallelogram of U, and so, if (ω, ω') is an appropriately chosen base for U it follows that $(2\omega - \omega', \omega + \omega')$ is a pair of primitive periods of f (g and h).

This observation leads to a more explicit construction of f, g and h. Let $\wp$ denote the Weierstraß P-function with a pair of primitive periods $2\omega - \omega'$ and $\omega + \omega'$. Then

$$F(z) = \frac{\big(\wp(z) - \wp(\omega/2)\big)\big(\wp(z) - \wp(3\omega/2)\big)^2}{\wp(z) - \wp(\omega)},$$

$$G(z) = F(z+\omega) \quad \text{and} \quad H(z) = F(z+\omega')$$

share the values 0, ∞, A and B, where $A/B \neq -1$ is a third root of -1.

2. Proof of Theorem 2

From a well known theorem of Rellich [7], p. 153, follows that the solutions of the differential equation $(dw/dz)^2 = 4w(w+1)(w-a)$ are either constants or elliptic functions. Thus, every solution of (3) is meromorphic in the plane, $U(z) = w(\gamma(z))$, where w is an appropriately chosen solution of the equation mentioned above.

We denote the solutions of $P(x,y) = 0$ by y_1, y_2 and y_3. It is easily seen that $x = 0,\ -1,\ a$ and ∞ are critical points, where y_1, y_2 and y_3 coincide (having values $0,\ -a,\ 1$ and ∞).

Let us consider the case $x = y = 0$ in more detail. The Newton–Puiseux diagram (see E. Hille [4], p. 105) looks very simple (see Figure 1).

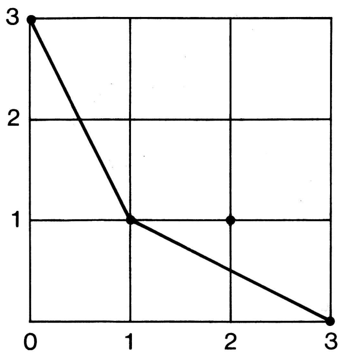

Figure 1. Newton-Puiseux diagram near $x = y = 0$.

Thus the solutions are

$$y_1(x) = \sum_{k=1}^{\infty} c_k x^{k/2}, \qquad y_2(x) = \sum_{k=1}^{\infty} (-1)^k c_k x^{k/2} \qquad (c_1 \neq 0) \tag{4}$$

and

$$y_3(x) = \sum_{k=2}^{\infty} d_k x^k \quad (d_2 \neq 0) \tag{4'}$$

which is regular since $y_1 y_2 y_3 = x^3$.

A similar behaviour is observed near the critical points ∞, a and -1. For the convenience of the reader we will write down the corresponding equations:

i) $x = y = \infty$ (set $x = 1/\xi, y = 1/\eta$):
$$\eta^3 - 3((a-1)\xi^2 - 2\xi)\eta^2 - 3(2\xi^2 - (\bar{a}-1)\xi)\eta - \xi^3 = 0.$$
ii) $x = a,\ y = 1$ (set $x = \xi + a, y = \eta + 1$) :
$$\eta^3 - 3((\bar{a}-1)\xi^2 - 2a\xi)\eta^2 - 3(2\bar{a}\xi^2 - (a-1)\xi)\eta - \xi^3 = 0.$$
iii) $x = -1,\ y = -a$ (set $x = \xi - 1, y = \eta - a$) :
$$\eta^3 - 3((\bar{a}-1)\xi^2 - 2\bar{a}\xi)\eta^2 - 3(2a\xi^2 - (a-1)\xi)\eta - \xi^3 = 0.$$

The discriminant $D(x)$ of our equation is given by

$$D(x) = \big(y_1(x) - y_2(x)\big)^2\big(y_2(x) - y_3(x)\big)^2\big(y_3(x) - y_1(x)\big)^2, \tag{5}$$

and from (4) and (4′) it is seen that $D(x) \sim \text{const.}\, x^3$ near $x = 0$, and, similarly, $D(x) \sim \text{const.}(x+1)^3$ near $x = -1$ and $D(x) \sim \text{const.}(x-a)^3$ near $x = a$.

However, at infinity, two of the solutions behave like $\pm\,\text{const.}\sqrt{x}$ while the third has a double pole. Thus, from (5) it follows that $D(x) \sim \text{const.}\, x^9$ as $x \to \infty$, and so the discriminant is

$$D(x) = \text{const.}\, x^3(x+1)^3(x-a)^3,$$

which shows that there are no critical points other than 0, ∞, a and -1.

Now, define locally

$$f = y_1 \circ U, \quad g = y_2 \circ U \quad \text{and} \quad h = y_3 \circ U. \tag{6}$$

The only critical singularities are the zeros, one-points, ($-a$)-points and poles of U.

As an example, consider a zero z_0 of U. Since U has only zeros of even multiplicity (which is two if z_0 is not a zero of γ'), it follows that $\sqrt{U}$ is holomorphic near z_0, and so from (4) and (4′) it follows that f, g and h are regular at z_0. A similar reasoning applies at the other points, and so, by the monodromy theorem, f, g and h are meromorphic in the finite plane.

Now it is easy to show that they share the values 1, 0, ∞ and $-a$. For example, let z_0 be a zero of f, say. Then, since f is a solution of (1) we must have $U(z_0) = 0$, which implies that every solution of (1) is zero, that is, $g(z_0) = h(z_0) = 0$.

Thus, Theorem 2 is completely proved.

3. Remarks on the proof of Theorem 1

The main tool will be Nevanlinna's theory of meromorphic functions, and it is assumed that the reader is familiar with its basic notations and results (see W.K. Hayman [3]).

Our hypothesis is that

(H) three distinct nonconstant meromorphic functions in the plane share four values.

For technical reasons it is sometimes convenient to denote our functions f, g and h also by f_1, f_2 and f_3. This should not give rise to any confusion (the patient reader is asked to look at (f_1, f_2, f_3) as a permutation of (f, g, h)).

For the same reason the shared values are denoted by a_1, a_2, a_3, a_4 and are assumed to be finite in the first part of the proof.

Only in the final part of the proof we will use the normalization $a_1 = -a$, $a_2 = 1$, $a_3 = 0$ and $a_4 = \infty$ which is then more convenient.

a) In Section 4, Cartan's auxiliary function (7) is used to show that for an arbitrary c-point z_0 (c is one of the shared values) with 'few' exceptions the following is true: z_0 is a simple c-point of two of the functions, while it is a multiple c-point of the third one.

b) In Sections 5–7, we will show that every derivative f_j' $(1 \le j \le 3)$ has 'many' zeros, most of them having multiplicity $k_j - 1 \ge 1$. Using various auxiliary functions we will be able to prove that (k_1, k_2, k_3) is a solution of the diophantic equation

$$\sqrt{xyz} = \sqrt{x} + \sqrt{y} + \sqrt{z} + 2,$$

which has exactly one solution $x = y = z = 4$.

c) In Section 8, the shared values are assumed to be $-a$, 1, 0 and ∞. With the aid of a) and b) we will be able to show that there are meromorphic third roots U, V and W of the functions $\prod_{j=1}^{3} f_j$, $\prod_{j=1}^{3}(f_j - 1)$ and $\prod_{j=1}^{3}(f_j + a)$, which are connected via $V = \alpha U - \alpha\bar{\alpha}$ and $W = \beta U + \bar{\beta} a$ $(\alpha^3 = \beta^3 = 1)$. Finally, it turns out that $a = \alpha \ne 1$. From this, (1), (2) and (3) will easily follow.

Besides the standard notation of Nevanlinna theory we use the abbreviations

$$T(r) = \max_{j=1}^{3} T(r, f_j)$$

and

$$\bar{N}_c(r) = \bar{N}\left(r, \frac{1}{f_j - c}\right)$$

for $c = a_1, a_2, a_3, a_4$.

Here, f_1, f_2 and f_3 denote the functions sharing the values a_1, a_2, a_3 and a_4, which are assumed to be finite throughout Sections 4–7.

We write $\sigma(r) \le o(T(r))$ as $r \to \infty$ to indicate that, given $\varepsilon > 0$, there is a set $E_\varepsilon \subset (0,\infty)$ of finite measure such that $\sigma(r) \le \varepsilon T(r)$ outside E_ε.

If F is a meromorphic function and p is a positive integer, then $\bar N(r,F,p)$ denotes the counting function of the p-fold poles of F, each pole counted simply (as indicated by $^-$). We will say that almost every pole of F has multiplicity p if $\bar N(r,F) = \bar N(r,F,p) + o(T(r))$. Note that this does not imply $N(r,F) = p\bar N(r,F,p) + o(T(r))$!

4. Cartan's auxiliary function

We will first collect in Lemma 1 some well known facts which are even true if two functions share four values.

Lemma 1 ([2], p. 547, [6], p. 373). *Under the hypothesis* (H) *the following is true:*

(a)
$$T(r,f_j) = \big(1+o(1)\big)T(r);$$

(b)
$$m\left(r,\frac{1}{f_j-f_k}\right) + N_1\left(r,\frac{1}{f_j-f_k}\right) = o\big(T(r)\big), \qquad j \ne k;$$

(c)
$$\sum_{\nu=1}^{4} N_1\left(r,\frac{1}{f_j-a_\nu}\right) = N\left(r,\frac{1}{f_j'}\right) + o\big(T(r)\big);$$

(d)
$$\sum_{\nu=1}^{4} \bar N\left(\frac{1}{f_j-a_\nu}\right) = \big(2+o(1)\big)T(r).$$

Proof. From Nevanlinna's second main theorem and our hypothesis (H) follows

$$\begin{aligned}\big(2+o(1)\big)T(r) &\le \sum_{\nu=1}^{4} \bar N\left(r,\frac{1}{f_j-a_\nu}\right) \le \bar N\left(r,\frac{1}{f_j-f_k}\right)\\ &\le T(r,f_j) + T(r,f_k) + O(1) \le \big(2+o(1)\big)T(r)\end{aligned}$$

($j \ne k$). Thus, the inequality sign must hold, and (a)–(d) are easy consequences (note that in the third inequality the term $m(r,1/(f_j-f_k)) + N_1(r,1/(f_j-f_k))$ has been omitted).

In [1] H. Cartan considered the auxiliary function

$$\varphi_a = \left(\frac{h'}{h-a} - \frac{g'}{g-a}\right) f + \left(\frac{f'}{f-a} - \frac{h'}{h-a}\right) g + \left(\frac{g'}{g-a} - \frac{f'}{f-a}\right) h, \tag{7}$$

where a is one of the shared values and where we have written f, g and h instead of f_1, f_2 and f_3. The estimate

$$T(r, \varphi_a) \le (3 + o(1))T(r) \tag{8}$$

follows immediately from the usual rules, and it is also seen very easily that φ_a vanishes at least twice at every common a_k-point, $a_k \neq a$. Thus, if

$$\varphi_a \not\equiv 0 \tag{9}$$

is taken for granted, we have

$$2 \sum_{a_k \neq a} \bar{N}_{a_k}(r) \le N\left(r, \frac{1}{\varphi_a}\right) \le (3 + o(1))T(r), \tag{10}$$

and a symmetry argument shows that

$$6 \sum_{k=1}^{4} \bar{N}_{a_k}(r) \le (12 + o(1))T(r) \tag{11}$$

is true, compatible with Lemma 1 (d).

Now it is stated in [1] that φ_a vanishes also in the common a-points, and if this were true we were allowed to replace the constant 6 in (11) by 7 and would so derive a contradiction to Lemma 1 (d).

However, φ_a does not necessarily vanish at the common a-points. (This has been kindly pointed out to me by Erwin Mues to whom I also would like to express my thanks for valuable discussions on this topic a few years ago (then we believed that Cartan's theorem is true).) For, if z_0 is a common a-point of f, g and h of multiplicity k, l and m, say, then from (7) it follows that

$$\varphi_a(z_0) = (m - l)f'(z_0) + (k - m)g'(z_0) + (l - k)h'(z_0), \tag{12}$$

and so $\varphi_a(z_0) = 0$ exactly in the following cases (up to permutations):

(i) $k = l = m = 1$,

(ii) $\min(k, l, m) > 1$,

(iii) $k = l > m = 1$, and

(iv) $m > k = l = 1$ and $f'(z_0) = g'(z_0)$.

On the other hand, $\varphi_a(z_0) \neq 0$ if, for example, $m > k = l = 1$ and $f'(z_0) \neq g'(z_0)$.

Before proving (9) we will first draw the consequences from the fact that φ_a vanishes with 'few' exceptions only in the common a_j-points, $a_j \neq a$.

Lemma 2. *Under the hypothesis* (H) *the following is true:*

(a) $$\bar{N}_a(r) = \left(\tfrac{1}{2} + o(1)\right) T(r) \qquad \text{for } a = a_1, a_2, a_3, a_4.$$

(b) *Almost every a-point ($a = a_1, \dots, a_4$) is a simple a-point of two of the functions and a multiple a-point of the third one.*

(c) $$\sum_{j=1}^{3} \bar{N}_1\left(r, \frac{1}{f_j - a}\right) = \left(\tfrac{1}{2} + o(1)\right) T(r) \qquad \textit{for } a = a_1, a_2, a_3, a_4.$$

(d) $$\sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f_j'}\right) = (2 + o(1)) T(r).$$

Proof. We remark that (9) is assumed for $a = a_1, a_2, a_3, a_4$ and will be proved later.

We already know that in (11) and consequently in (10) the equality sign must hold, from which assertion (a) follows.

(b) is an immediate consequence of our considerations of Cartan's ingeneously chosen auxiliary function (7). From this and (a) also (c) and (d) follow, since to almost every a-point there corresponds a zero of exactly one of the derivatives f_j'.

Lemma 3. *Under the hypothesis* (H) *we have $\varphi_a \not\equiv 0$ for $a = a_1, a_2, a_3 a_4$.*

Proof. Assume $\varphi_a = 0$ for some $a = a_j$. It is the same to say that

(13′) $$H'(F - G) + G'(H - F) + F'(G - H) \equiv 0,$$

where $F = (f - a)^{-1}$, $G = (g - a)^{-1}$ and $H = (h - a)^{-1}$. Differentiating this identity yields

(13″) $$H''(F - G) + G''(H - F) + F''(G - H) \equiv 0,$$

while

(13) $$H(F - G) + G(H - F) + F(G - H) \equiv 0$$

holds true trivially. Since the differences $F - G$, $H - F$ and $G - H$ do not vanish identically, the linear homogeneous system (13–13″) has a nontrivial solution for all but countably many z. Thus the Wronskian of H, G and F must vanish identically and so H, G and F are linearly dependent:

(14) $$\psi_a := \frac{\lambda}{f - a} + \frac{\mu}{g - a} + \frac{\nu}{h - a} \equiv 0$$

for some nontrivial choice of λ, μ, ν. Since at least one of the other values $a_k \neq a$ is actually attained, we have

$$\lambda + \mu + \nu = 0. \tag{15}$$

It is clear that ψ_b cannot vanish for every $b = a_1, \ldots, a_4$, thus there is at least one b with $\psi_b \not\equiv 0$. The estimate

$$T(r, \psi_b) \leq 3T(r) - 2\bar{N}_b(r) + o\big(T(r)\big)$$

follows immediately from the definition of ψ_b, and from (15) it follows that ψ_b vanishes in every a_k-point, $a_k \neq b$, and even with multiplicity two at least if $a_k \neq a$. This gives

$$\bar{N}_a(r) + 2 \sum_{a_k \neq a, b} \bar{N}_{a_k}(r) \leq N\left(r, \frac{1}{\psi_b}\right) \leq 3T(r) - 2\bar{N}_b(r) + o\big(T(r)\big)$$

or, equivalently,

$$2 \sum_{k=1}^{4} \bar{N}_{a_k}(r) \leq \big(3 + o(1)\big)T(r) + \bar{N}_a(r). \tag{16}$$

This is even stronger than (10) which holds if $\varphi_a \not\equiv 0$. Adding up these inequalities for varying a, we get

$$7 \sum_{k=1}^{4} \bar{N}_{a_k}(r) \leq \big(12 + o(1)\big)T(r) + \sum_{\varphi_a \not\equiv 0} \bar{N}_a(r).$$

This, together with Lemma 1 (d), leads to $\bar{N}_a(r) = o\big(T(r)\big)$ if $\varphi_a \equiv 0$, while from (16) and Lemma 1 (d) $T(r)\big(1 + o(1)\big) \leq \bar{N}_a(r)$ follows, which is obviously absurd. Hence, Lemma 3 is proved.

5. Three auxiliary functions

We put $\Pi(w) = \prod_{\nu=1}^{4} (w - a_\nu)$ and consider the auxiliary functions

$$F_j = \frac{f_j'}{\Pi(f_j)} \cdot \frac{(f_j - f_{j+1})(f_j - f_{j+2})}{f_{j+1} - f_{j+2}} \tag{17}$$

($j = 1, 2, 3$), where $f_4 = f_1$ and $f_5 = f_2$. The usual rules for the proximity function give

$$m(r, F_j) \leq m\left(r, \frac{f_j'}{\Pi(f_j)}\right) + 2m(r, f_j) + m(r, f_{j+1}) + m(r, f_{j+2})$$
$$+ m\left(r, \frac{1}{f_{j+1} - f_{j+2}}\right) + O(1).$$

Here the last relevant term is $o(T(r))$ by Lemma 1 (b), while the first term on the right hand side is $o(T(r))$ by Nevanlinna's lemma on the logarithmic derivative (note that $f'/\Pi(f) = \sum_{\nu=1}^{4} A_\nu f'/(f-a_\nu)$). Also by Lemma 1 (d) and the second main theorem we have $m(r, f_k) = o(T(r))$ ($k = 1,2,3$). On the other hand, the construction of F_j is made in such a way that the zeros and poles of the nominator and the denominator cancel out by Lemma 1 up to a sequence having counting function $o(T(r))$. Thus, F_j is a small function:

$$T(r, F_j) = o(T(r)). \tag{18}$$

According to Lemma 2 (d), at least one of the derivatives f_j' has 'many' zeros in the sense that

$$\bar{N}\left(r, \frac{1}{f_j'}\right) \neq o(T(r)). \tag{19}$$

Thus, there exists an integer $k_j \geq 2$ such that

$$\bar{N}\left(r, \frac{1}{f_j'}, k_j - 1\right) \neq o(T(r)). \tag{20}$$

Lemma 4. *Under the hypothesis* (20) *there exists a small meromorphic function* κ_j,

$$T(r, \kappa_j) = o(T(r)), \tag{21}$$

such that

$$k_j \frac{F_{j+1}}{F_j} = (1+\kappa_j)^2, \quad k_j \frac{F_{j+2}}{F_j} = \left(1 + \frac{1}{\kappa_j}\right)^2, \quad \frac{F_{j+1}}{F_{j+2}} = \kappa_j^2. \tag{22}$$

Moreover, in almost every zero z_0 *of* f_j' *of multiplicity* $k_j - 1$ *we have*

$$f_{j+1}' + \kappa_j f_{j+2}' = 0 \qquad (z = z_0). \tag{23}$$

We remark that the last equation (22), which is independent of k_j, shows that there are at most two distinct integers $k_j \geq 2$ having property (20). If there are two, the corresponding functions are κ_j and $-\kappa_j$.

Proof of Lemma 4. We may assume $j = 1$. Let z_0 be a zero of f_1' of multiplicity $k_1 - 1$ such that $f_1(z_0) \in \{a_1, \ldots, a_4\}$ (note that almost every zero of f_1' has the last property). Then from (17) follows

$$\frac{k_1 F_2}{F_1} = \left(\frac{f_2'}{f_3'} - 1\right)^2, \quad \frac{k_1 F_3}{F_1} = \left(1 - \frac{f_3'}{f_2'}\right)^2, \quad \frac{F_2}{F_3} = \left(\frac{f_2'}{f_3'}\right)^2$$

at $z = z_0$, and elimination of f_2'/f_3' yields

$$4\frac{F_2}{F_3} = \left(k_1\frac{F_2}{F_1} - 1 - \frac{F_2}{F_3}\right)^2 \tag{24}$$

and

$$2\frac{f_2'}{f_3'} = \frac{F_2}{F_3} + 1 - k_1\frac{F_2}{F_1} \tag{25}$$

at $z = z_0$.

From hypothesis (20), however, it follows that (24) holds throughout the whole plane (a function, like the difference of the left hand side and the right hand side of (24), with characteristic $o(T(r))$ may not have zeros with counting function $\neq o(T(r))$, until it vanishes identically). Thus, (22) and (23) follow if we put

$$2\kappa_1 = k_1\frac{F_2}{F_1} - 1 - \frac{F_2}{F_3},$$

while (21) is a consequence of (18).

We may assume that the statement of Lemma 4 is true for $j = 1$, and we want to show that it is true for $j = 1, 2$ and 3. This will be done in two steps.

Lemma 5. *For $j = 1, 2, 3$ and every $k = 2, 3, 4, \ldots$ we have*

$$\bar{N}\left(r, \frac{1}{f_j'}, k-1\right) \leq \left(\frac{2}{3} + o(1)\right) T(r). \tag{26}$$

Proof. Again we may assume $j = 1$. Since (26) is trivial if (20) is false ($j = 1$), we may assume that the statement of Lemma 4 holds for $j = 1$ and $k_j = k$.

Consider the auxiliary function

$$F_a := (f_1, f_2, a, f_3) = \frac{f_1 - a}{f_2 - a} : \frac{f_1 - f_3}{f_2 - f_3},$$

where $a \in \{a_1, \ldots, a_4\}$ is arbitrary. We note that the multiple zeros of $f_1 - a$, $f_2 - a$ and $f_3 - a$ are zeros, poles and 1-points of F_a, thus

$$T(r, F_a) \geq \left(\frac{1}{6} + o(1)\right) T(r) \tag{27}$$

by Lemma 2 (c). On the other hand,

$$\begin{aligned} T(r, F_a) &\leq m\left(r, \frac{1}{f_2 - a}\right) + N_1\left(r, \frac{1}{f_2 - a}\right) + o(T(r)) \\ &= T(r) - \bar{N}_a(r) + o(T(r)) = \left(\tfrac{1}{2} + o(1)\right) T(r). \end{aligned}$$

From (23) ($j = 1$) follows that in almost every a_ν-point z_0 ($a \neq a_\nu$) of order k_1

$$F_a(z_0) = 1 + \kappa_1(z_0).$$

Since, by (27), $F_a \not\equiv 1 + \kappa_1$, this gives

$$\begin{aligned}\sum_{a_j \neq a} \bar{N}\left(r, \frac{1}{f_1 - a_j}, k_1\right) &\leq N\left(r, \frac{1}{F_a - 1 - \kappa_1}\right) + o(T(r)) \\ &\leq T(r, F_a) + o(T(r)) \leq \left(\tfrac{1}{2} + o(1)\right) T(r).\end{aligned}$$

Adding up these inequalities for $a = a_1, a_2, a_3, a_4$, inequality (26) follows.

6. Zeros of the derivatives

As already mentioned we want to prove that (19) and so Lemma 4 is true for $j = 1, 2, 3$ (and appropriate k_j), that is

Lemma 6. *For $j = 1, 2, 3$ we have*

$$\bar{N}\left(r, \frac{1}{f_j'}\right) \neq o(T(r)).$$

Proof. We may assume that the statement of Lemma 6 is true for $j = 1$, and, if it is false in general, it is false for $j = 3$.

If it is false for $j = 2$, too, we derive a contradiction as follows:

Since $\bar{N}(r, 1/f_1') \leq N(r, 1/f_1') \leq T(r, f_1') \leq (2 + o(1))T(r)$, we have by Lemma 2(d)

$$\begin{aligned}\bar{N}\left(r, \frac{1}{f_1'}, 1\right) &= \bar{N}\left(r, \frac{1}{f_1'}\right) + o(T(r)) \\ &= \sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f_j'}\right) + o(T(r)) = (2 + o(1))T(r),\end{aligned}$$

which contradicts Lemma 5 ($j = 1, k = 2$).

Thus, if Lemma 6 is false in general, we may assume that it is false for $j = 3$ and true for $j = 1$ and 2.

First case: Assume that, for $j = 1, 2$, there is exactly one integer k_j with property (19). Then we find, using Lemma 5,

$$\sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f_j'}\right) = \sum_{j=1}^{2} \bar{N}\left(r, \frac{1}{f_j'}\right) + o(T(r)) \leq \left(\frac{4}{3} + o(1)\right) T(r)$$

in contrast to Lemma 2 (d).

Second case: There exist, for $j = 1$, say, two different integers having property (19), denoted by k_1 and l_1 ($k_1 < l_1$). As already mentioned, we have

$$(22') \qquad l_1 \frac{F_2}{F_1} = (1-\kappa_1)^2, \quad l_1 \frac{F_3}{F_1} = \left(1 - \frac{1}{\kappa_1}\right)^2, \quad \frac{F_2}{F_3} = \kappa_1^2,$$

and k_1 and l_1 are uniquely determined.

The corresponding integer k_2 will be chosen minimal (if not uniquely determined).

The trivial observations

$$(2+o(1))T(r) \geq N\left(r, \frac{1}{f_2'}\right) \geq (k_2 - 1)\bar{N}\left(r, \frac{1}{f_2'}\right) + o(T(r))$$

and

$$(2+o(1))T(r) \geq$$
$$(k_1 - 1)\bar{N}\left(r, \frac{1}{f_1'}\right) + (l_1 - k_1)\bar{N}\left(r, \frac{1}{f_1'}, l_1 - 1\right) + o(T(r))$$

lead together with Lemma 2 (d) to the inequality (note that $\bar{N}(r, 1/f_3') = o(T(r))$)

$$\frac{2}{k_1 - 1} + \frac{2}{k_2 - 1} \geq 2 + \alpha \frac{l_1 - k_1}{k_1 - 1},$$

where α is some positive constant. Thus we have

$$(28) \qquad \min(k_1, k_2) = 2 \qquad \text{and} \qquad l_1 > k_1.$$

From the first equation of (22), $j = 1$, and of (22′) follows

$$(29) \qquad \frac{k_1}{l_1} = \left(\frac{1+\kappa_1}{1-\kappa_1}\right)^2 \qquad \text{and so} \qquad \kappa_1 < 0,$$

and from (22), $j = 1, 2$, follows

$$\kappa_1 = -(\sqrt{k_1 k_2} \pm 1)/(\sqrt{k_1} \mp 1).$$

This leads to the diophantic equation

$$(30) \qquad \pm 4\sqrt{k_1 k_2} \pm 4\sqrt{k_1} + 2(k_1 + l_1)\sqrt{k_2} = (l_1 - k_1)(k_2 + 1) - 4.$$

Multiplying by $\sqrt{k_1}$ and $\sqrt{k_2}$, respectively, we get two additional diophantic equations

$$(30') \qquad 2(k_1+l_1)\sqrt{k_1k_2}+\big(4-(l_1-k_1)(k_2+1)\big)\sqrt{k_1}\pm 4k_1\sqrt{k_2}=\mp 4k_1$$

and

$$(30'') \qquad \pm 4\sqrt{k_1k_2}\pm 4k_2\sqrt{k_1}+\big(4-(l_1-k_1)(k_2+1)\big)\sqrt{k_2}=-2(l_1+k_1)k_2.$$

Now (30)–(30″) may be regarded as a system of linear equations with solution $(\sqrt{k_1k_2},\sqrt{k_1},\sqrt{k_2})$. The determinant D of this system, however, is given by

$$\begin{aligned}\pm\tfrac{1}{4}D &= 4k_2(k_1+l_1)^2+16k_1(1-k_2)\\ &\quad+\big((l_1-k_1)(k_2+1)-4\big)\big((l_1-k_1)k_2+5l_1+3k_1-4\big),\end{aligned}$$

which is easily seen to be positive. Thus, $\sqrt{k_1}$, $\sqrt{k_2}$ and $\sqrt{k_1k_2}$ are rational numbers in contrast to (29). This proves Lemma 6.

7. Another diophantic equation

From equations (22), $1\le j\le 3$, it is posssible to eliminate the functions F_1, F_2 and F_3. We obtain

$$(31) \qquad k_1=\kappa_3^2(\kappa_1+1)^2,\quad k_2=\kappa_1^2(\kappa_2+1)^2\quad\text{and}\quad k_3=\kappa_2^2(\kappa_3+1)^2$$

under the constraints

$$(32) \qquad (\kappa_1\kappa_2\kappa_3)^2=1\qquad\text{and}\qquad \min(k_1,k_2,k_3)\ge 2.$$

We note that κ_1, κ_2 and κ_3 are constants, because no κ_j assumes the values -1, 0 and ∞, but this is of no importance here.

Again, elimination of κ_1, κ_2 and κ_3 leads to the diophantic equation

$$(33) \qquad \alpha\beta\gamma\delta\sqrt{k_1k_2k_3}-(\alpha\sqrt{k_1}+\beta\sqrt{k_2}+\gamma\sqrt{k_3}+1+\delta)=0,$$

where α, β, γ and δ are allowed to assume the values ±1 independently of each other.

If the left hand side is denoted by $\alpha\beta\gamma\delta H(k_1,k_2,k_3)$, then H is strictly increasing in all its variables in the range $k_j\ge 2$, and so $H(2,2,2)\le 0$ gives

$$2\sqrt{2}\le\sqrt{2}(\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta)+\alpha\beta\gamma\delta+\alpha\beta\gamma,$$

which is only possible if $\alpha=\beta=\gamma=\delta=1$. Thus, only the diophantic equation

$$(34) \qquad \sqrt{xyz}-\sqrt{x}-\sqrt{y}-\sqrt{z}-2=0$$

has to be considered in the range $x,y,z\ge 2$.

We remark that $\alpha=\beta=\gamma=\delta=1$ corresponds to $\sqrt{k_1}=\kappa_3(\kappa_1+1)$, $\sqrt{k_2}=\kappa_1(\kappa_2+1)$, and $\sqrt{k_3}=\kappa_2(\kappa_3+1)$ and $\kappa_1\kappa_2\kappa_3=1$.

Lemma 7. *The diophantic equation* (34) *has exactly one solution in the range* $x, y, z \geq 2 : x = y = z = 4$.

Remark. It then follows easily that $\kappa_1 = \kappa_2 = \kappa_3 = 1$.

Proof. Let (x, y, z) be a solution. By symmetry we may assume $2 \leq x \leq y \leq z$. To obtain an upper bound for y we use the monotonicity of H, which implies $0 = H(x, y, z) \geq H(2, y, y)$ and so $\sqrt{2y^2} - \sqrt{2} - 2\sqrt{y} - 2 \leq 0$. This gives the upper bound

$$y \leq \left(\frac{\sqrt{2}}{2} + \sqrt{\frac{3}{2} + \sqrt{2}} \right)^2 < 5.83 \qquad \text{and so} \qquad y \leq 5.$$

Now (34) is equivalent with

$$z = \left(\frac{\sqrt{x} + \sqrt{y} + 2}{\sqrt{xy} - 1} \right)^2 .$$

In the range $2 \leq x \leq y \leq 5$, however, z is never an integer except when $x = y = 4$ (the distance of z to the next integer is at least 0.17, except when $x = y = 4$). This proves Lemma 7.

8. Summary

The quintessence of what we have proved in Sections 4–7 is that almost every common c-point z_0 ($c \in \{a_1, \ldots, a_4\}$) is a simple c-point of two of the functions and a c-point of multiplicity 4 of the third one. Moreover, the sum of derivatives $f' + g' + h'$ vanishes at z_0 (we prefer now to write f, g and h instead of f_1, f_2 and f_3).

We will call such a c-point 'regular'; the sequence of 'irregular' c-points has counting function $o(T(r))$.

By the Möbius transform $w \mapsto (w, a_2, a_3, a_4)$ the values a_2, a_3, a_4 are mapped onto 1, 0, ∞, while a_1 corresponds to $-a = (a_1, a_2, a_3, a_4) \neq 1, 0, \infty$.

It is convenient to change notations and to assume that

(H*) f, g and h share the values 1, 0, ∞ and $-a$.

We set

(35) $$\Phi = fg + gh + hf, \qquad \Psi = f + g + h$$

and

(36) $$U_w = (f - w)(g - w)(h - w) \qquad \text{for} \quad w = 0, 1, -a.$$

The meromorphic function $\Phi - \Psi$ has poles of order at most 4 in regular poles and zeros of order at least 2 in regular zeros and 1-points, while U_0U_1 has poles of order 12 in regular poles and zeros of order 6 in regular zeros and 1-points. The other poles of $\Phi - \Psi$ and zeros of U_0U_1 form a sequence having counting function $o(T(r))$.

Thus, $(\Phi - \Psi)^3/U_0U_1$ is a small function (its characteristic is $o(T(r))$) which assumes the value 27 in all regular $(-a)$-points. Since the counting function of these points is $\left(\frac{1}{2} + o(1)\right) T(r)$,

$$(\Phi - \Psi)^3 = 27U_0U_1 \tag{37a}$$

follows.

In the same manner we find

$$(\Phi + a\Psi)^3 = 27U_0U_{-a}, \tag{37b}$$

$$\left(\Phi + (a-1)\Psi - 3a\right)^3 = 27U_1U_{-a} \tag{37c}$$

and

$$\left(3U_0 + (a-1)\Phi - a\Psi\right)^3 = 27U_0U_1U_{-a}. \tag{37d}$$

Hence, there are meromorphic functions U, V and W such that $U^3 = U_0$, $V^3 = U_1$ and $W^3 = U_{-a}$, which are uniquely determined up to a factor $\sqrt[3]{1}$. Thus,

$$\Phi - \Psi = 3UV, \tag{38a}$$

$$\Phi + a\Psi = 3UW, \tag{38b}$$

$$\Phi + (a-1)\Psi - 3a = 3VW \tag{38c}$$

and

$$3U^3 + (a-1)\Phi - a\Psi = 3\omega UVW, \qquad \omega^3 = 1, \tag{38d}$$

follows from (37 a–d) for suitably chosen branches of U, V and W, while

$$V^3 = U^3 - \Phi + \Psi - 1 \tag{39a}$$

and

$$W^3 = U^3 + a\Phi + a^2\Psi + a^3 \tag{39b}$$

are consequences of (36).

Now (38a) and (39a) give $V^3 - U^3 + 3UV + 1 = 0$, hence

$$V = \alpha U - \bar{\alpha}, \tag{40a}$$

and, similarly, (38b) and (38c) yield

$$W = \beta U + \bar{\beta} a, \tag{40b}$$

where α and β are third roots of unity.

Since $\Phi = 3U(aV + W)/(a+1)$ and $\Psi = 3U(W - V)/(a+1)$, equations (38c), (40a) and (40b) lead to an algebraic equation

$$\begin{aligned} &\big[\beta a(1-\alpha) + \alpha(1-\beta)\big]U^2 + \big[a^2\bar{\beta}(1-\alpha) + a(\bar{\alpha}\beta - \alpha\bar{\beta}) + \bar{\alpha}(\beta - 1)\big]U \\ &\qquad + a(a+1)(\bar{\alpha}\bar{\beta} - 1) = 0 \end{aligned} \tag{41}$$

for $U = U(z)$. Since U is nonconstant, (41) must be trivial, and this leads to

$$\alpha\beta - 1 = (\alpha - 1)(a^2 + a + 1) = a(\bar{\alpha} - 1) + (\alpha - 1) = 0. \tag{42}$$

(The same result is derived if equation (38d) is used instead of (38c).)

If $\alpha = 1$ we have $\beta = 1$ and so

$$\Psi = f + g + h = 3U,$$

which is impossible, since, in a regular pole, Ψ has a pole of order 4, while U has a double pole there. Thus, $\alpha \neq 1$ and so

$$a = \alpha \neq 1 \text{ is a third root of unity.}$$

Now consider

$$\begin{aligned} (w-f)(w-g)(w-h) &= w^3 - \Psi w^2 + \Phi w - U^3 \\ &= w^3 - 3\big((\bar{a}-1)U^2 - 2U\big)w^2 - 3\big(2U^2 - (a-1)U\big)w - U^3 \equiv P(U,w). \end{aligned}$$

Then, $w = f, g, h$ are solutions of $P\big(U(z), w\big) = 0$, as stated in Theorem 1 (formulae (1) and (2)).

What is left is to show that U is a solution of equation (3). To this end we define the function γ_0 by

$$\gamma_0 := \frac{U'^2}{U(U+1)(U-a)}.$$

Then $\gamma_0 = 4\gamma'^2$ and γ' is entire if we are able to show that U has only zeros, (-1)-points, a-points and poles of even order.

For example, let z_0 be a zero of U. From $P(U,f) = P(U,g) = P(U,h) = 0$ then follows that exactly two of our functions f, g, h behave like $\text{const.}\, U^{1/2}$ near z_0 (see the proof of Theorem 2), which is only possible if z_0 is a zero of U of even order.

A similar reasoning applies in poles, a-points and (-1)-points of U (one has to use equations i), ii) and iii) in the proof of Theorem 2).

This proves Theorem 1 completely.

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