Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 14, 1989, 89–101

RIEMANN SURFACES WITH THE AD-MAXIMUM PRINCIPLE

Pentti Järvi

Introduction

Let W be an open Riemann surface. We say that W satisfies the (absolute) AD-maximum principle if every end V of W, i.e., a subregion of W with compact relative boundary ∂V, has the property that each function in $AD(\bar{V})$, the class of analytic functions with a finite Dirichlet integral on $\bar{V} = V \cup \partial V$, assumes its supremum on ∂V. It is natural to expect that the validity of this principle presumes some sort of weakness of the ideal boundary of W. Actually, our main theorem (Theorem 1) asserts that, given any end $V \subset W$ and any $f \in AD(\bar{V})$, the cluster set of f attached to the relative ideal boundary of V is a null-set of class N_D in the familiar notation of Ahlfors–Beurling. This result is completely analogous to that of Royden concerning Riemann surfaces which satisfy a similar principle for bounded analytic functions [19].

The interest in the class of surfaces introduced above stems in part from the fact that it contains both $\mathcal{O}_{KD}$ and $\mathcal{O}_{A^\circ D}$ (see Chapter 2). In particular, owing to Theorem 1, the boundary theorems of Constantinescu (on $\mathcal{O}_{KD}$-surfaces) and Matsumoto (on $\mathcal{O}_{A^\circ D}$-surfaces) can be given a unified treatment. In fact, our version improves these results in three respects. First, it applies to a wider class of surfaces. Second, instead of AD-functions we deal with the larger class of meromorphic functions with a finite spherical Dirichlet integral. Third, we will show that the behavior of the functions at the ideal boundary is not just continuous but even "analytic" in a well justified sense of the word. This in turn makes it possible to draw certain conclusions of an algebraic nature (see Theorem 3 and its corollary).

The main theorem also bears on Royden's version of the Riemann–Roch theorem (on $\mathcal{O}_{KD}$-surfaces). It turns out that, roughly speaking, his result is the pullback of the classical case via a finite sheeted covering map. Furthermore, Theorem 1 entails a Kuramochi-type result concerning the nonexistence of certain meromorphic functions on Riemann surfaces with arbitrary "holes" (Theorem 6).

doi:10.5186/aasfm.1989.1413

1. The main theorem

Let V be an end of an open Riemann surface W. For the sake of convenience, we always assume that ∂V consists of a finite number of piecewise analytic closed curves. Let f be a nonconstant analytic function on $\bar{V}$. Assuming that $z \in \mathbf{C} \setminus f(\partial V)$, the index of z is defined by

$$i(z) = (2\pi)^{-1} \int_{\partial V} d \arg\bigl(f(p) - z\bigr).$$

With suitable interpretation (see [19]), $i(z)$, as well as the valence $v(z)$ of f at z with respect to $\bar{V}$, can be defined also for $z \in f(\partial V)$ and even in such a way that the expression $\delta(z) = i(z) - v(z)$ remains unaltered whenever V is subjected to a compact modification.

Assume now that W satisfies the AD-maximum principle, and let $f \in AD(\bar{V})$ be nonconstant. In what follows, our principal aim is to show that $\delta(z) \geq 0$ for all $z \in \mathbf{C}$ and the set $E = \bigl\{z \in \mathbf{C} \mid \delta(z) > 0\bigr\}$ is of class N_D. The proof is largely based on the ideas of Royden [19]. However, there are some extra problems due to the fact that the class AD is not closed under composition of functions. This state of affairs explains the division of the proof into "topological" and "analytical" parts. More precisely, we first show that E is totally disconnected and then, by means of this preliminary result, that E actually belongs to N_D. We begin with a simple lemma.

Lemma. *Let $K \subset \mathbf{C}$ be a proper continuum with connected complement. Then $\mathbf{C} \setminus K$ carries a nonconstant analytic function g such that both g and g' are bounded.*

Proof. Obviously we may assume that K is nowhere dense in $\mathbf{C}$. Fix two distinct points $z_1, z_2 \in K$. Denote by φ_1 the restriction to $\mathbf{C} \setminus K$ of a linear fractional mapping that sends z_1 to 0 and z_2 to ∞. Further, denote by φ_2 some branch of the mapping $z \mapsto z^{1/2}$, $z \in \varphi_1(\mathbf{C} \setminus K)$. Pick out a point $z_3 \in \mathbf{C} \setminus \overline{\varphi_2\bigl(\varphi_1(\mathbf{C} \setminus K)\bigr)}$, and let φ_3 stand for the inversion $z \mapsto 1/(z - z_3)$. Finally, let φ_4 denote the map $z \mapsto z^2$ and φ_5 the map $z \mapsto \bigl(z - (1/z_3)^2\bigr)^2$. Then $g = \varphi_5 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ is the desired function. □

We return to the function $f \in AD(\bar{V})$ fixed previously. Set $N = \max\bigl\{i(z) \mid z \in \mathbf{C}\bigr\}$ and let E_k denote the closed set $\bigl\{z \in \mathbf{C} \mid \delta(z) = i(z) - v(z) \geq k\bigr\}$, $k \in Z$. We claim that $E_0 = \mathbf{C}$ and E_1 $(= E)$ is totally disconnected. Observing that E_{N+1} is empty, we assume that E_{k+1} is totally disconnected, $k \leq N$. We show that E_k also is totally disconnected provided that it is nowhere dense in $\mathbf{C}$. Set $D_k = E_k \setminus E_{k+1}$ and let $z \in D_k$. Since δ remains unaltered by the removal from V of a compact set with nice boundary, we may modify V so that z has a neighborhood U such that no point of $U \cap D_k$ is assumed (on $\bar{V}$) by f, while each point of $U \setminus D_k$ is assumed (on V) by f. Furthermore, we may arrange so that $U \cap f(\partial V) = \emptyset$. If $U \cap D_k$ is not totally disconnected, we can obviously find in $U \cap D_k$ a proper continuum K with connected complement. Let g be a function described in the preceding lemma. Because g' is bounded, $g \circ f$ belongs to $AD(\bar{V})$. However, since g assumes a larger value at some point of U than its maximum in $\mathbf{C} \setminus U$, $g \circ f$ must take a larger value in V than on ∂V. This violates the AD-maximum principle for W. Thus $U \cap D_k$ is totally disconnected. Since this is true for each $z \in D_k$ and E_{k+1} is totally disconnected, we conclude that E_k is totally disconnected.

Assume then that D_k has interior points, and suppose the interior of D_k has a boundary point z in the complement of E_{k+1}. Modifying V suitably, we can find an open disc U containing z such that $U \cap f(\partial V) = \emptyset$, f assumes no values in $U \cap D_k$ and assumes all values in $U \setminus D_k$. Since $U \cap D_k$ has interior points, we can find a rational function g whose only pole is in the interior of $U \cap D_k$ and which is larger in U than in $\mathbf{C} \setminus U$. Since g' is bounded in $\mathbf{C} \setminus (U \cap D_k)$, $g \circ f \in AD(\bar{V})$. However, $|g(f(q))| > \max\{|g(f(p))| \mid p \in \partial V\}$ for each $q \in V$ with $f(q) \in U$. Thus we have again arrived at a contradiction to the AD-maximum principle for W. It follows that D_k has no boundary points in the complement of E_{k+1}. But this implies that $D_k = \mathbf{C} \setminus E_{k+1}$. In other words, D_k is the whole complement of E_{k+1} provided it contains interior points.

Since f is bounded in $\bar{V}$, D_0 contains a neighborhood of ∞. Therefore, D_k has interior points only if $k = 0$. Hence $E_0 = \mathbf{C}$ and the deficiency set E $(= E_1)$ is totally disconnected. We conclude that f has bounded valence: $v(z) \leq N$ for all $z \in \mathbf{C}$.

We are now in a position to establish the definitive result $E \in N_D$. The proof is again by induction. Recalling that E_{N+1} is empty, assume $E_{k+1} \in N_D$ for some $k \leq N$. Let $z \in D_k = E_k \setminus E_{k+1}$. As above, we may modify V so that z has a neighborhood U such that $U \cap f(\partial V) = \emptyset$ and no point of $U \cap D_k$ is assumed by f, while each point of $U \setminus D_k$ is assumed by f. Suppose there is a compact part, say F, of $U \cap D_k$ that does not belong to N_D. Then there is a nonconstant AD-function g defined in $\mathbf{C} \setminus F$. As shown previously, f has bounded valence, so that $g \circ f \in AD(\bar{V})$. By the maximum principle, g assumes a larger value at some point of U than its maximum in $\mathbf{C} \setminus U$. Hence $\sup\{|g(f(p))| \mid p \in \bar{V}\} > \max\{|g(f(p))| \mid p \in \partial V\}$, in violation of the AD-maximum principle for W. We conclude that the compact parts of $U \cap D_k$ are of class N_D. Since this holds for each $z \in D_k$, and $E_{k+1} \in N_D$ also, we infer that $E_k \in N_D$. It follows that E belongs to N_D as was asserted. We have thereby completed the proof of

Theorem 1. *Let W be an open Riemann surface satisfying the AD-maximum principle, and let V be an end of W. Let $f \in AD(\bar{V})$ be nonconstant. Then f has bounded valence; in fact, $v(z) \leq i(z)$ for each $z \in \mathbf{C}$. Moreover, the deficiency set $E = \{z \in \mathbf{C} \mid v(z) - i(z) < 0\}$ is of class N_D.*

2. Some consequences

2.1. We begin with some notation and terminology. By definition, a harmonic function u with a finite Dirichlet integral on a Riemann surface W is in $KD(W)$ if $*du$ is semiexact, i.e., $\int_\gamma *du = 0$ for every dividing cycle γ on W. If $KD(W)$ reduces to the constants, W is said to belong to $\mathcal{O}_{KD}$ [21, p. 132]. Further, W belongs to $\mathcal{O}_{A^\circ D}$ [21, p. 17] provided every bordered subregion V of W, with compact or noncompact border ∂V, has the property that the double of $(V, \partial V)$ about ∂V belongs to $\mathcal{O}_{AD}$, the class of surfaces without nonconstant AD-functions. Both $\mathcal{O}_{KD}$ and $\mathcal{O}_{A^\circ D}$ provide examples of surfaces with the AD-maximum principle as appears from

Proposition 1. (a) *Every Riemann surface in* $\mathcal{O}_{KD} \cup \mathcal{O}_{A^\circ D}$ *satisfies the* AD*-maximum principle.*

(b) *Let* W *be a Riemann surface satisfying the* AD*-maximum principle. Then* W *belongs to* $\mathcal{O}_{AD}$.

Proof. For $W \in \mathcal{O}_{A^\circ D}$ the validity of the AD-maximum principle is essentially proved in [21, pp. 373–4]. For $W \in \mathcal{O}_{KD}$ the corresponding statement readily follows from assertion IV in [5, p. 1995] (see also [22, p. 254]). Assertion (b) is of course trivial. □

Let V be an end of W. Then $MC(V)$ denotes the class of meromorphic functions on V which have a limit at every point of the relative Stoïlow ideal boundary β_V of V. Furthermore, $BV(V)$ stands for the class of constants and of meromorphic functions of bounded valence on V, while $MD^*(V)$ denotes the class of meromorphic functions with a finite spherical Dirichlet integral on V. Whenever f is a function of class MC, we let f^* denote the extension of f to the (relative) ideal boundary.

We first show that the MD^*-functions behave continuously at the ideal boundary provided W satisfies the AD-maximum principle.

Theorem 2. *Let* W *be a Riemann surface satisfying the* AD*-maximum principle, and let* V *be an end of* W. *Then* $MD^*(\bar{V}) = BV(\bar{V}) \subset MC(\bar{V})$. *Furthermore,* $f^*(\beta_V)$ *belongs to* N_D *for every* $f \in MD^*(V)$.

Proof. Suppose first that $f \in AD(\bar{V})$. By Theorem 1 f has bounded valence and the deficiency set E belongs to N_D. It is not difficult to verify that $\mathrm{Cl}(f;\beta_V)$, the cluster set of f attached to β_V, is contained in E (details can be found in [10, p. 303]). Since E is totally disconnected, each $\mathrm{Cl}(f;p)$, the cluster set attached to $p \in \beta_V$, must be a singleton, i.e., $f \in MC(\bar{V})$. Finally, $f^*(\beta_V) \subset E$ implies $f^*(\beta_V) \in N_D$.

Now let $f \in MD^*(\bar{V})$ be nonconstant. Let $V_1, \ldots, V_n$ be mutually disjoint subends of V such that $\bar{V} \setminus \left(\bigcup_{i=1}^n V_i\right)$ is compact and f omits in $\cup_{i=1}^n \bar{V}_i$ a compact set $E \subset \mathbf{C}$ of positive area measure. By a theorem of Nguyen Xuan Uy [23, Theorem 4.1], we can find a nonconstant analytic function g such that both g

and g' are bounded in $\mathbf{C}\setminus E$. Fix $i\in\{1,\dots,n\}$. We are going to show that $h = g\circ(f\mid \bar V_i)$ belongs to $AD(\bar V_i)$. To this end, choose $R>0$ such that $E\subset D(0,R) = \{z\in\mathbf{C}\mid |z|<R\}$. Set $F_1 = \bar V_i\cap f^{-1}(D(0,R))$, $F_2 = \bar V_i\cap f^{-1}(\hat{\mathbf{C}}\setminus D(0,R))$ ($\hat{\mathbf{C}} = \mathbf{C}\cup\{\infty\}$) and choose $M>0$ such that $|g'(z)|\le M$ for $z\in\mathbf{C}\setminus E$. Then

$$\begin{aligned}\iint_{F_1} dh\wedge * d\bar h &= \iint_{F_1} |g'(f(p))|^2 df\wedge * d\bar f\\ &\le (1+R^2)^2M^2\iint_{F_1}\frac{1}{(1+R^2)^2}df\wedge * d\bar f\\ &\le (1+R^2)^2M^2\iint_{F_1}\frac{1}{(1+|f(p)|^2)^2}df\wedge * d\bar f<\infty.\end{aligned}$$

Let φ denote the mapping $z\mapsto 1/z$, $z\in\mathbf{C}\setminus D(0,R)$. Then $g\mid \overline{\hat{\mathbf{C}}\setminus D(0,R)} = g_1\circ\varphi$ with g_1 analytic in $\overline{D(0,1/R)}$. Suppose $|g_1'(z)|\le M_1$ for $\in \overline{D(0,1/R)}$. Then

$$\begin{aligned}\iint_{F_2} dh\wedge * d\bar h &= \iint_{F_2} |g_1'(\varphi(f(p)))|^2 d(\varphi\circ f)\wedge * d\overline{(\varphi\circ f)}\\ &\le M_1^2\iint_{F_2} d(\varphi\circ f)\wedge * d\overline{(\varphi\circ f)}\\ &\le M_1^2(1+1/R)^2)^2\iint_{F_2}\frac{1}{\left(1+|\varphi(f(p))|^2\right)^2}d(\varphi\circ f)\wedge * d\overline{(\varphi\circ f)}\\ &= M_1^2(1+1/R)^2)^2\iint_{F_2}\frac{1}{(1+|f(p)|^2)^2}df\wedge * d\bar f<\infty.\end{aligned}$$

Thus $h\in AD(\bar V_i)$. By Theorem 1, $h = g\circ(f\mid \bar V_i)$ has bounded valence. Of course, the same is true of $f|\bar V_i$. Since $i\in\{1,\dots,n\}$ was arbitrary and $\bar V\setminus\cup_{i=1}^n V_i$ is compact, f has bounded valence, too.

Now pick out a point $z_0\in\mathbf{C}$ such that the valence of f attains its maximum at z_0. Choose a small disc U centered at z_0 such that $f^{-1}(U)$ consists of a finite number of mutually disjoint Jordan domains in V. Let ψ stand for the mapping $z\mapsto 1/(z-z_0)$. Then $\psi\circ f$ is bounded in $\bar V\setminus f^{-1}(U)$. This implies that $\psi\circ f\in AD(\bar V\setminus f^{-1}(U))$. By the first part of the proof, $\psi\circ f\in MC(\bar V\setminus f^{-1}(U))$ and $(\psi\circ f)^*(\beta_V)\in N_D$. But then $f\in MC(\bar V)$ and $f^*(\beta_V)\in N_D$ also.

The inclusion $BV(\bar V)\subset MD^*(\bar V)$ being trivial, the proof is complete. □

Remark. The argument proving the relation $h\in AD(\bar V)$ also appears in the forthcoming paper [12].

In view of the next theorem we may even claim that the MD^*-functions admit an analytic extension to the ideal boundary.

Theorem 3. *Let W be a Riemann surface satisfying the AD-maximum principle and let p_0 be a point of β, the ideal boundary of W, such that some end $V \subset W$ with $p_0 \in \beta_V$ carries a nonconstant MD^*-function. Then there is an end $V_0 \subset W$ with $p_0 \in \beta_{V_0}$ and an analytic map φ of bounded valence from $\bar{V}_0$ into the closed unit disc $\bar{D}$ with $\varphi(\partial V_0) \subset \partial \bar{D}$ such that for every $f \in MD^*(\bar{V}_0)$ $(= BV(\bar{V}_0))$ there is a unique function g meromorphic on $\bar{D}$ so that $f = g \circ \varphi$. In particular, $g \mapsto g \circ \varphi$ is an isomorphism of $M(\bar{D})$, the field of meromorphic functions on $\bar{D}$, onto $MD^*(\bar{V}_0)$.*

We omit the proof because it is essentially the same as that of [10, Theorem 5], given in the context of Riemann surfaces with the AB-maximum principle. How to treat sets of class N_D instead of N_B appears from [9, p. 14].

Corollary. *Let W be a Riemann surface satisfying the AD-maximum principle and let $V \subset W$ be an end. Then $MD^*(\bar{V})$ $(= BV(\bar{V}))$ is a field.*

Remark. In view of Proposition 1, Theorems 2 and 3 provide improvements of the boundary theorems of Constantinescu ([5, Théorème 1], [6, Theorem], [22, Theorem X 4 C (a)]) and Matsumoto ([15, Theorem 3], [21, Theorem VI 2 C]; see also [12, Theorem 3]). In particular, it follows from Theorem 2 that the requirement that the boundary elements be weak is superfluous in Constantinescu's theorem. This statement is at odds with Remark 2 in [22, p. 265]. It seems that the authors of [22] overlooked the possibility that the class of functions involved in Theorem X 4 C (c) reduces to the constants.

2.2. A global counterpart to the preceding theorem is the following

Theorem 4. *Let W be a Riemann surface satisfying the AD-maximum principle. Then either*
(a) *$MD^*(W) = BV(W) = \mathbf{C}$, or*
(b) *$MD^*(W)(= BV(W))$ is a field algebraically isomorphic to the field of rational functions on a compact Riemann surface W_0, which is uniquely determined up to a conformal equivalence. Moreover, the isomorphism is induced by an analytic mapping φ of bounded valence from W into W_0 such that the deficiency set of φ (i.e., the set of points in W_0 not covered maximally) is of class N_D.*

Proof. Suppose that $MD^*(W)$ contains a nonconstant function f. By Theorem 2, $f^*(\beta)$ belongs to N_D. Thus $f^*(\beta)$ does not separate the plane. Hence the valence of f is finite and constant in $\hat{\mathbf{C}} \setminus f^*(\beta)$; in other words, the deficiency set of f belongs to N_D. The theorem now follows from [9, Theorem 7]. □

Remark. Suppose, in particular, that W has finite genus. Then W can be taken as the complement of a closed set of class N_D on a compact surface W^*. It is readily shown (see e.g. [9, Lemma 6]) that the elements of $BV(W)$ coincide with the restrictions to W of the rational functions on W^*. Accordingly, the same is true of $MD^*(W)$ so that we may take $W_0 = W^*$ in this situation.

The next theorem shows how to find the genus of W_0 in terms of analytic differentials on W. The proof to be given is an adaptation of the argument by Accola [1, pp. 23–24]. As usual, $\Gamma_a(W)$ denotes the space of square integrable analytic differentials on W, while $\Gamma'_a(W)$ stands for the subspace of $\Gamma_a(W)$ consisting of differentials which are exact outside some compact set (which may depend on the differential).

Theorem 5. *Let W be a Riemann surface satisfying the AD-maximum principle and suppose $MD^*(W)$ contains a nonconstant function. Let W_0 be the compact Riemann surface described in Theorem 4. Then the genus of W_0 equals* $\dim \Gamma'_a(W)$.

Proof. Let $\varphi\colon W \to W_0$ be the mapping given in the preceding theorem. Since the genus of W_0 equals $\dim \Gamma_a(W_0)$, it is enough to show that $\Gamma'_a(W) = \{\varphi^*\omega \mid \omega \in \Gamma_a(W_0)\}$, where $\varphi^*\omega$ denotes the pullback of ω via φ.

Let $E \subset W_0$ denote the deficiency set of φ; recall that E is of class N_D. Since E is totally disconnected, we can find an open simply connected neighborhood U of E. Fix $\omega \in \Gamma_a(W_0)$. Then $\omega \mid U = df$ with f analytic in U. Now $\varphi^{-1}(W_0 \backslash U) \subset W$ is compact and $\varphi^*\omega \mid \varphi^{-1}(U) = d(f \circ \varphi)$, so that $\varphi^*\omega \in \Gamma'_a(W)$.

Conversely, fix $\omega \in \Gamma'_a(W)$ not identically zero (if $\Gamma'_a(W) = \{0\}$, then also $\Gamma_a(W_0) = \{0\}$ by the preceding argument). By definition, there is a compact set $K \subset W$ such that $\omega \mid W \backslash K = df$ for some $f \in AD(W \backslash K)$. Further, let ω_0 be a nontrivial meromorphic differential on W_0 whose poles do not lie in E. As above, we can find a compact set $K' \subset W$ and a function $g \in AD(W \setminus K')$ such that $\varphi^*\omega_0 \mid W \setminus K' = dg$. We are going to show that the function $\omega/\varphi^*\omega_0$ belongs to $MD^*(W)$.

Fix $p_0 \in \beta$, the ideal boundary of W. Invoking Theorem 3, we can find an end $V_0 \subset W$ such that $p_0 \in \beta_{V_0}$ and $\bar{V}_0 \cap (K \cup K') = \emptyset$ and an analytic mapping ψ from $\bar{V}_0$ into $\bar{D}$ such that $f \mapsto f \circ \psi$ is an isomorphism of $M(\bar{D})$ onto $MD^*(\bar{V}_0)$. Hence there are functions $f_0, g_0 \in M(\bar{D})$ such that $f \mid \bar{V}_0 = f_0 \circ \psi$ and $g \mid \bar{V}_0 = g_0 \circ \psi$. Of course, f'_0/g'_0 also belongs to $M(\bar{D})$, and because $(f'_0/g'_0) \circ \psi = (df/dg) \mid \bar{V}_0 = (\omega/\varphi^*\omega_0) \mid \bar{V}_0$, $(\omega/\varphi^*\omega_0) \mid \bar{V}_0$ belongs to $MD^*(\bar{V}_0)$. By the compactness of β, $\omega/\varphi^*\omega_0 \in MD^*(W)$, as was asserted.

By Theorem 4, there is a rational function h_0 on W_0 such that $\omega/\varphi^*\omega_0 = h_0 \circ \varphi$. Thus $\omega = (h_0 \circ \varphi)\varphi^*\omega_0 = \varphi^*(h_0\omega_0)$, i.e., ω is the pullback via φ of an analytic differential on W_0. The proof is complete. □

2.3. Consider now the situation of [17, Section 3] (see also [21, pp. 138–144]); in other words, suppose W is an open Riemann surface of class $\mathcal{O}_{KD}$, and let $\mathcal{M}(W)$ denote the class of all meromorphic functions f on W such that f has a finite number of poles and a finite Dirichlet integral over the complement of a neighborhood of its poles. Assume also that $\mathcal{M}(W)$ is nontrivial, i.e., contains a nonconstant function. Since W satisfies the AD-maximum principle (Proposition 1), each $f \in \mathcal{M}(W)$ is bounded outside a compact subset of W. Thus $\mathcal{M}(W)$ constitutes a ring. We maintain that the quotient field of $\mathcal{M}(W)$ is $MD^*(W)$ $(= BV(W))$. Indeed, given a nonconstant function $f \in MD^*(W)$ choose a point $z_0 \in \mathbf{C}$ such that the valence of f attains its maximum at z_0. Then $g = 1/(f - z_0)$ is bounded off a compact subset of W (cf. the proof of Theorem 2). Hence $g \in \mathcal{M}(W)$. Since $f = (1 + z_0 g)/g$, the assertion follows.

By Theorem 4, there exist a compact Riemann surface W_0 and an analytic mapping φ of a bounded valence from W into W_0 such that each $f \in MD^*(W)$ admits a representation

$$f = g \circ \varphi, \tag{$*$}$$

where g is a rational function on W_0. Clearly, $f \in \mathcal{M}(W)$ if and only if g is a rational function on W_0 whose poles lie outside the deficiency set of φ. It follows that the problem of whether there is a function in $\mathcal{M}(W)$ which is a multiple of a given divisor and has a given principal part can be decided in terms of analytic objects on W_0. In this sense, Theorem 2 in [17, p. 47] ([21, Theorem II 16 I]) can be regarded as the pullback via the map φ of the classical Riemann–Roch theorem. Of course, the rigidity of the class $\mathcal{M}(W)$, as evidenced in $(*)$, imposes severe limitations on potential singularities for elements of $\mathcal{M}(W)$. In particular, $\mathcal{M}(W)$ separates points of W if and only if W has finite genus. As observed previously, W is then the complement of a set of class N_D on a compact Riemann surface.

Remark 1. In his paper [1] Accola discusses more broadly generalizations of some classical theorems from the point of view of Heins' composition theorem [8], which is a special case of Theorem 4.

Remark 2. A frequent substitute for the field of rational functions is the class of quasirational functions in the sense of Ahlfors [3. p. 316]. In the present situation a function is quasirational if and only if it belongs to $MD^*(W)$ and is bounded away both from 0 and from ∞ outside some compact subset of W.

2.4. Let U_S denote the class of open Riemann surfaces whose ideal boundary contains a point of positive harmonic measure [21, p. 385]. Suppose that W belongs to U_S and satisfies the AD-maximum principle, and let K be an arbitrary compact set in W with connected complement. Let $f \in MD^*(W \setminus K)$. By Theorem 2, f admits a continuous extension to the ideal boundary of W. On the other hand, Theorem X 4 C (c) in [22] implies that f must be constant. Thus, denoting by $\mathcal{O}_{MD^*}$ the class of Riemann surfaces without nonconstant MD^*-functions, we obtain the following theorem, which contains [21, Theorem VI 5 B] and [12, Corollary to Theorem 3].

Theorem 6. *Let W be a Riemann surface which satisfies the AD-maximum principle and belongs to U_S, and let K be an arbitrary compact set in W with connected complement. Then $W \setminus K \in \mathcal{O}_{MD^*}$.*

Familiar instances of U_S-surfaces are furnished by the interesting class $\mathcal{O}_{HD} \setminus \mathcal{O}_G$, where $\mathcal{O}_{HD}$ is the class of Riemann surfaces without nonconstant Dirichlet bounded harmonic functions and $\mathcal{O}_G$ the class of parabolic surfaces. Recall that the ideal boundary of each $W \in \mathcal{O}_{HD} \setminus \mathcal{O}_G$ contains exactly one point of positive harmonic measure. Since every surface in $\mathcal{O}_{HD}$ also satisfies the AD-maximum principle (for $\mathcal{O}_{HD} \subset \mathcal{O}_{KD}$), we have the original version of Kuramochi [14, Theorem 1], [21, Corollary to Theorem III 5I].

Corollary. *Let $W \in \mathcal{O}_{HD} \setminus \mathcal{O}_G$ and let K be an arbitrary compact set in W with connected complement. Then $W \setminus K \in \mathcal{O}_{AD}$.*

3. Characterizations of Riemann surfaces with the AD-maximum principle

3.1. It is clear that a Riemann surface of finite genus satisfies the AD-maximum principle if and only if it belongs to $\mathcal{O}_{AD}$. On the other hand, it is known that for these surfaces $\mathcal{O}_{AD} = \mathcal{O}_{KD}$ [21, Theorem II 14 D]. However, in the general case the inclusions given in Proposition 1 are strict. The next theorem gives some criteria to recognize surfaces with the AD-maximum principle. In particular, we will show that it suffices to impose the maximum principle on the bounded AD-functions. Given an open Riemann surface W and an end $V \subset W$, we set $ABD(V) = \{f \mid f \text{ is bounded in } V \text{ and } f \in AD(V)\}$ and say that W satisfies the ABD-maximum principle if $\max\{|f(p)| \mid p \in \partial V\} = \sup\{|f(p)| \mid p \in \bar{V}\}$ for each end $V \subset W$ and for each $f \in ABD(\bar{V})$.

Theorem 7. *Let W be an open Riemann surface. Then the following statements are equivalent:*

(1) *W satisfies the AD-maximum principle.*
(2) *W satisfies the ABD-maximum principle.*
(3) *$MD^*(\bar{V}) \subset BV(\bar{V})$ for every end $V \subset W$.*
(4) *For every end $V \subset W$ and for every $f \in MD^*(\bar{V})$ the cluster set of f attached to the relative ideal boundary of V is totally disconnected.*

Proof. The implications (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are direct consequences of Theorem 2. Also, (4) $\Rightarrow$ (3) is immediate by observing that the valence function is finite and constant in every component of $\hat{\mathbf{C}} \setminus \big(f(\partial V) \cup \mathrm{Cl}(f;\beta_V)\big)$. The implication (1) $\Rightarrow$ (2) being trivial, there remains to be proved (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (2): Suppose there is an end $V \subset W$ and an ABD-function f on $\bar{V}$ with $\max\{|f(p)| \mid p \in \partial V\} < \sup\{|f(p)| \mid p \in \bar{V}\}$. We may assume that

$\sup\{|f(p)| \mid p \in \bar{V}\} = 1$. We maintain that $AD(\bar{V})$ contains a function of unbounded valence. If $f \notin BV(\bar{V})$, there is nothing to prove. Otherwise pick out a sequence of points (z_n) in $f(V)$ such that

$$\sum_{n=1}^{\infty} \log\left(\frac{1}{1-|z_n|}\right)^{-1/2} < \infty.$$

By a result of Carleson [4, Theorem 1], there is a nonconstant AD-function g in the open unit disc such that $g(z_n) = 0$ for each n. Since f has bounded valence, $g \circ f \in AD(\bar{V})$, while $g \circ f \notin BV(\bar{V})$. The implication follows.

(2) $\Rightarrow$ (1): Suppose that W satisfies the ABD-maximum principle, and let V be an end of W. If suffices to show that $AD(\bar{V})$ contains no unbounded function. Assume $AD(\bar{V})$ contains one, say f, and fix $R > 0$ such that $f(\partial V) \subset D(0,R)$ ($=$ the open disc of radius R centered at 0). Assume first that the interior of $(\hat{\mathbf{C}} \setminus D(0,R)) \setminus f(V)$ is nonempty. Then we can find a point z_0 in this interior and a positive r such that $D(z_0,r) \cap f(V) \neq \emptyset$ and $D(z_0,r) \cap D(0,R) = \emptyset$. It follows that $1/(f - z_0)$ belongs to $ABD(\bar{V})$ and takes a larger value in V than on ∂V. This contradicts the ABD-maximum principle for W. Assume then that $f(V)$ is dense in $\hat{\mathbf{C}} \setminus \overline{D(0,R)}$. Since $f \in AD(\bar{V})$, f omits in $\bar{V}$ a compact set $E \subset \mathbf{C} \setminus \overline{D(0,R)}$ of positive area measure. Invoking [23, Theorem 4.1], we find a nonconstant analytic function g such that g and g' are bounded in $\mathbf{C} \setminus E$. Then $g \circ f \in ABD(\bar{V})$. The set $f(V)$ being dense in $\hat{\mathbf{C}} \setminus \overline{D(0,R)}$, $g \circ f$ assumes a larger value at some point of V than its maximum on ∂V. Thus we have again arrived at a contradiction to the ABD-maximum principle for W. □

Remark. In view of Theorem 2 and Corollary to Theorem 3 one may ask whether the condition $MD^*(\bar{V}) \subset MC(\bar{V})$ or the field property of $MD^*(\bar{V})$ implicates the validity of the AD-maximum principle. Cf. also [11, Theorems 7 and 2]. Unfortunately, we have not been able to answer these questions.

3.2. Suppose V is an end of a Riemann surface satisfying the AD-maximum principle. If the genus of V is infinite, $AD(\bar{V})$ fails to separate points of $\bar{V}$; this is immediate by Theorem 3. In other words, in case $AD(\bar{V})$ separates points, V can be taken as the complement on a finite Riemann surface of a closed set of class N_D. Actually, in order to establish this result one need not the full force of the AD-maximum principle as shown by

Theorem 8. *Let W be an open Riemann surface, and let $V \subset W$ be an end such that ∂V consists of a finite number of closed analytic curves. Suppose that $AD(\bar{V})$ separates the points of $\bar{V}$ and for each $f \in AD(\bar{V})$ $\max\{|f(p)| \mid p \in \partial V\} = \sup\{|f(p)| \mid p \in \bar{V}\}$. Then there exist a finite Riemann surface V^* and a compact set $E \subset V^*$ of class N_D such that V is conformally equivalent to $V^* \setminus E$. Further, V^* is uniquely determined up to a conformal equivalence.*

Proof. The assertion is an easy consequence of a theorem by Royden. Namely, by [20, Theorem 3] V has finite genus. Therefore, V can be taken as a subdomain of a compact surface W_0 such that $W_0 \setminus V$ consists of a finite number of mutually disjoint closed discs $U_1, \ldots, U_n$ corresponding the components of ∂V and of a closed set E. Set $V^* = V \cup E$. Assuming that E fails to be of class N_D, we can find a nonconstant function f in $AD(W_0 \setminus E)$ [21, Theorem I 8 E]. By the maximum principle $\max\{|f(p)| \mid p \in \bigcup_{i=1}^n \partial U_i\} < \sup\{|f(p)| \mid p \in V\}$, contrary to the assumption. The uniqueness of V^* is obtained by observing that sets of class N_D are removable singularities for conformal mappings. □

Remark. Wermer [24] has proved a similar result about Riemann surfaces satisfying the corresponding maximum principle for bounded analytic functions.

4. Concluding remarks

Needless to say, the validity of the AD-maximum principle is preserved under conformal mappings. More generally, given two Riemann surfaces W and W' and a proper analytic mapping $W \to W'$, W satisfies the AD-maximum principle if and only if W' does. Also it is clear from the very definition that validity of the AD-maximum principle, unlike belonging to $\mathcal{O}_{AD}$, is a property of the ideal boundary (see [21, p. 54]).

Riemann surfaces with small boundary are close to being maximal. For instance, it is known that surfaces of class $\mathcal{O}_{KD}$ or $\mathcal{O}_{A^\circ D}$ are essentially maximal, i.e., they cannot be realized as nondense subdomains of other surfaces. For $\mathcal{O}_{KD}$ this result is due to Jurchescu (see [22, p. 270]) and for $\mathcal{O}_{A^\circ D}$ to Qiu Shuxi [16, Theorem 3]. On the other hand, one can exhibit Riemann surfaces which are essentially extendable and satisfy the AD-maximum principle; take, for example, the construction by Heins [7, pp. 298–299] modified in an obvious way. However, provided that a surface also carries enough locally defined MD^*-functions, it is essentially maximal; what is more, the ideal boundary is absolutely disconnected (see [22, pp. 240 and 270]).

Proposition 2. *Let W be a Riemann surface which satisfies the AD-maximum principle and let β be the ideal boundary of W. Suppose that for every point $p \in \beta$ there is an end $V \subset W$ with $p \in \beta_V$ such that $MD^*(V)$ contains a nonconstant function. Then β is absolutely disconnected.*

Proof. Fix $p_0 \in \beta$. By Theorem 3 we can find an end $V \subset W$ with $p_0 \in \beta_V$ and a function $f \in BV(\bar{V}) = MD^*(\bar{V})$ such that $f(V) \subset D$, the open unit disc, and $f(\partial V) \subset \partial D$. Since $f^*(\beta_V)$ belongs to N_D (Theorem 2), and the mapping $p \mapsto f(p)$, $V \setminus f^{-1}(f^*(\beta_V)) \to D \setminus f^*(\beta_V)$ is proper, we can apply [13, Theorem 1]. Thus β_V is absolutely disconnected. Since $p_0 \in \beta$ was arbitrary, the proposition follows. □

References

[1] ACCOLA, R.: Some classical theorems on open Riemann surfaces. - Bull. Amer. Math. Soc. 73, 1967, 13–26.

[2] AHLFORS, L., and A. BEURLING: Conformal invariants and function-theoretic null-sets. - Acta Math. 83, 1950, 101–129.

[3] AHLFORS, L., and L. SARIO: Riemann surfaces. - Princeton University Press, Princeton, N.J., 1960.

[4] CARLESON, L.: On the zeros of functions with bounded Dirichlet integrals. - Math. Z. 56, 1952, 289–295.

[5] CONSTANTINESCU, C.: Sur le comportement d'une fonction analytique à la frontière idéale d'une surface de Riemann. - C. R. Acad. Sci. Paris 245, 1957, 1995–1997.

[6] CONSTANTINESCU, C.: On the behavior of analytic functions at boundary elements of Riemann surfaces. - Rev. Roumaine Math. Pures Appl. 2 (Hommage à S. Stoïlow pour son 70e anniversaire), 1957, 269–276 (Russian).

[7] HEINS, M.: Riemann surfaces of infinite genus. - Ann. of Math. 55, 1952, 296–317.

[8] HEINS, M.: On certain meromorphic functions of bounded valence. - Rev. Roumaine Math. Pures Appl. 2 (Hommage à S. Stoïlow pour son 70e anniversaire), 1957,263–267.

[9] JÄRVI, P.: Removability theorems for meromorphic functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 12, 1977, 1–33.

[10] JÄRVI, P.: Meromorphic functions on certain Riemann surfaces with small boundary. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980. 301–315.

[11] JÄRVI, P.: On meromorphic functions continuous on the Stoïlow boundary. - Ann. Acad. Sci. Fenn. Ser. A I Math. 9, 1984, 33–48.

[12] JÄRVI, P.: On the continuation of meromorphic functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 12, 1987, 177–184.

[13] JURCHESCU, M.: A maximal Riemann surface. - Nagoya Math. J. 20, 1962, 91–93.

[14] KURAMOCHI, Z.: On the behavior of analytic functions on abstract Riemann surfaces. - Osaka Math. J. 7, 1955, 109–127.

[15] MATSUMOTO, K.: Analytic functions on some Riemann surfaces II. - Nagoya Math. J. 23, 1963, 153–164.

[16] QIU, S.: The essential maximality and some other properties of a Riemann surface of $\mathcal{O}_{A^\circ D}$. - Chin. Ann. Math. Ser. B 6, 1985, 317–323.

[17] ROYDEN, H.L.: The Riemann–Roch theorem. - Comment. Math. Helv. 34, 1960, 37–51.

[18] ROYDEN, H.L.: On a class of null-bounded Riemann surfaces. - Comment. Math. Helv. 34, 1960, 52–66.

[19] ROYDEN, H.L.: Riemann surfaces with the absolute AB-maximum principle. - Proceedings of the Conference on Complex Analysis, Minneapolis, 1964, edited by A. Aeppli, E. Calabi and H. Röhrl. Springer-Verlag, Berlin–Heidelberg–New York, 1965, 172–175.

[20] ROYDEN, H.L.: Algebras of bounded analytic functions on Riemann surfaces. - Acta Math. 114, 1965, 113–142.

[21] SARIO, L., and M. NAKAI: Classification theory of Riemann surfaces. - Die Grundlehren der mathematischen Wissenschaften 164. Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[22] SARIO, L., and K. OIKAWA: Capacity functions. - Die Grundlehren der mathematischen Wissenschaften 149. Springer-Verlag, Berlin–Heidelberg–New York, 1969.

[23] UY, N.X.: Removable sets of analytic functions satisfying a Lipschitz condition. - Ark. Mat. 17, 1979, 19–27.

[24] WERMER, J.: The maximum principle for bounded functions. - Ann. of Math. 69, 1959, 598–604.

University of Helsinki
Department of Mathematics
Hallituskatu 15
SF-00100 Helsinki
Finland

Received 4 December 1987