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L^p-AVERAGING DOMAINS AND THE POINCARÉ INEQUALITY

Susan G. Staples*

1. Introduction

We shall assume in this paper that Ω and D are proper subdomains of $\mathbf{R}^n$ with $n \geq 2$. For such domains Ω and bounded D, we say that a function $u \in L^1_{\mathrm{loc}}(\Omega)$ is of *bounded mean oscillation in* Ω *with respect to* D, $u \in \mathrm{BMO}(\Omega, D)$, if $\|u\|_{*,D} < \infty$, where

$$\|u\|_{*,D} = \sup_{D' \subset \Omega} \frac{1}{m(D')} \int_{D'} |u - u_{D'}|\, dm.$$

Here D' is any domain in Ω obtained by a similarity transformation of D and $u_{D'}$ is the average of u over D', i.e.

$$u_{D'} = \frac{1}{m(D')} \int_{D'} u\, dm = \fint_{D'} u\, dm.$$

In the standard definition of BMO the supremum is taken either over balls or cubes in Ω. It is a well known fact that these two give equivalent norms in $\mathbf{R}^n$; that it is also true for arbitrary domains is an elementary consequence of the main result of this paper. We denote the usual BMO norm over balls by $\|u\|_{*,B^n} = \|u\|_*$. The question naturally arises for which types of domains D can we say

$$c_1 \|u\|_* \leq \|u\|_{*,D} \leq c_2 \|u\|_*, \tag{1.1}$$

where c_1 and c_2 are constants independent of u.

The first half of (1.1) holds for all bounded domains D. This fact follows directly from the inequality

$$\sup_{B \subset D} \frac{1}{m(B)} \int_B |u - u_B|\, dm \leq c \sup_{\tau B \subset D} \frac{1}{m(B)} \int_B |u - u_B|\, dm, \qquad c = c(n, \tau). \tag{1.2}$$

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Here the supremum on the right hand side is taken over balls B satisfying $\tau B \subset D$ for some constant $\tau > 1$ and c is a constant independent of u [RR].

We determine which domains D satisfy the latter half of (1.1) by reformulating this question more generally as a localization problem. Let D be a domain with $m(D) < \infty$. The bounded domains in question are precisely those for which $u \in \mathrm{BMO}(D)$ implies $u \in L^1(D)$ and the following inequality holds for $p = 1$:

$$(1.3) \qquad \left(\frac{1}{m(D)} \int_D |u - u_D|^p \, dm\right)^{1/p} \le c \left(\sup_{B \subset D} \frac{1}{m(B)} \int_B |u - u_B|^p \, dm\right)^{1/p},$$

where c is a constant independent of u and B is any ball in D. We say that D is an L^p*-averaging domain* if it satisfies (1.3). Bounded L^1-averaging domains then satisfy (1.1).

In Section 2 we characterize these domains in terms of the quasihyperbolic metric on D, $k(x, y)$. In particular, we show that for $p \ge 1$, D is an L^p-averaging domain if and only if $k(x, y)$ is L^p integrable over D. In the proof for sufficiency we actually prove the stronger result, namely that for each $\tau > 1$

$$(1.4) \qquad \left(\frac{1}{m(D)} \int_D |u - u_D|^p \, dm\right)^{1/p} \le c \left(\sup_{\tau B \subset D} \frac{1}{m(B)} \int_B |u - u_B|^p \, dm\right)^{1/p},$$

where $c = c(\tau)$. Since balls are L^p-averaging domains (1.2) is a special case of this result.

Properties and examples of L^p-averaging domains are examined in Section 3. For example, we show that L^p-averaging domains satisfy a Poincaré type inequality and that John domains are L^p-averaging domains. We also prove that the class of L^p-averaging domains is preserved under quasi-isometries, but not under quasiconformal maps.

We can regard the quantity

$$\frac{1}{m(D)} \int_D |u - u_D| \, dm = \operatorname*{avg\,osc}_D u$$

as a seminorm for u over D. The BMO seminorm

$$\sup_{B \subset D} \frac{1}{m(B)} \int_B |u - u_B| \, dm = \sup_{B \subset D} (\operatorname*{avg\,osc}_B u).$$

then gives a corresponding local seminorm. By definition L^1-averaging domains have the local to global norm property given by (1.3), namely

$$(1.5) \qquad \operatorname*{avg\,osc}_D u \le c \sup_{B \subset D} (\operatorname*{avg\,osc}_B u).$$

We can also ask which types of domains satisfy the same type of inequality for seminorms other than the one given by average oscillation. Gehring and Martio [GM] have answered this question for the Lipschitz class of mappings, denoted by $\mathrm{Lip}_\alpha(D)$.

A mapping $f\colon D \to \mathbf{R}^n$ is in $\mathrm{Lip}_\alpha(D)$, $0 < \alpha \le 1$ if for some $m < \infty$,

$$|f(x_1) - f(x_2)| \le m|x_1 - x_2|^\alpha \tag{1.6}$$

in D. The mapping f belongs to $\mathrm{loc\,Lip}_\alpha(D)$ if there exists a constant $m < \infty$ such that (1.6) holds whenever x_1 and x_2 lie in any open ball which is contained in D. The seminorms $\|f\|_\alpha$ and $\|f\|_\alpha^{\mathrm{loc}}$ are defined as the smallest m for which (1.6) holds in the respective sets. Gehring and Martio have characterized the domains which satisfy

$$\|f\|_\alpha \le c\,\|f\|_\alpha^{\mathrm{loc}}$$

for all $f \in \mathrm{loc\,Lip}_\alpha(D)$.

In Section 4 we consider the domains D which satisfy (1.5) with average oscillation replaced by the maximal oscillation of u over D, namely

$$\operatorname*{osc}_D u = \sup_D u - \inf_D u. \tag{1.7}$$

The corresponding local norm is given by the maximal oscillation of u over balls in D. We call domains satisfying

$$\operatorname*{osc}_D u \le c \sup_{B\subset D} (\operatorname*{osc}_B u) \tag{1.8}$$

oscillation domains.

Finally we study the geometry of such domains D. If D is an oscillation domain it is necessary that each point in the boundary of D is contained in the closure of some ball B in D. Moreover if each point in D lies in a ball $B \subset D$ of fixed radius $\delta > 0$ then D is an oscillation domain.

2. L^p-averaging domains

We establish some notation and definitions first. We let D denote a proper open subdomain of $\mathbf{R}^n$ with $m(D) < \infty$; here $m(D)$ is the n-dimensional Lebesgue measure of D, $d(z, \partial D)$ is the distance from z to the boundary of D, and dm represents Lebesgue measure. We let B and B' denote open balls in $\mathbf{R}^n$, ω_n the volume of the unit ball in $\mathbf{R}^n$, Q any cube in $\mathbf{R}^n$ and $l(Q)$ the side length of Q, and τB (similarly σQ) that ball with the same center as B and expanded by a constant factor of $\tau > 1$.

Definition. The quasihyperbolic distance between x and y in D is given by

$$k(x,y)=k(x,y;D)=\inf_{\gamma}\int_{\gamma}\frac{1}{d(z,\partial D)}\,ds,$$

where γ is any rectifiable curve in D joining x to y. Gehring and Osgood have proven that for any two points x and y in D there is a quasihyperbolic geodesic arc joining them [GO].

In this section we determine which domains satisfy (1.3).

2.1. Theorem. *If D is an L^p-averaging domain, then*

$$\left(\fint_D k(x,x_0)^p\,dm\right)^{1/p}\leq a \tag{2.2}$$

for each x_0 in D where a depends only on $d(x_0,\partial D)$, n, p, $m(D)$, and the constant c in (1.3).

Proof. Let B be any ball in D with center x_1 and radius r. We first show

$$\left(\fint_B k(x,x_1)^p\,dm\right)^{1/p}\leq \alpha, \tag{2.3}$$

where α is a constant depending only on n and p. We may assume that $x_1=0$. Then for each $x\in B$, $d(x,\partial D)\geq r-|x|$ and thus

$$\int_B k(x,0)^p\,dm\leq\int_B\left(\log\frac{r}{r-|x|}\right)^p\,dm.$$

Switching to polar coordinates and integrating one verifies (2.3). Let $u(x)=k(x,x_0)$. From the triangle inequality we obtain

$$\left(\fint_B |u(x)-u_B|^p\,dm\right)^{1/p}\leq 2\left(\fint_B |u(x)-u(x_1)|^p\,dm\right)^{1/p}\leq 2\alpha.$$

Since D is an L^p-averaging domain, we have

$$\left(\fint_D |u(x)-u_D|^p\,dm\right)^{1/p}\leq 2c\alpha. \tag{2.4}$$

Let $r=d(x_0,\partial D)$ and $B=B(x_0,r/2)$. Assume $u_D\geq 2$, and then on B,

$$u(x)\leq\log\frac{r}{r-|x-x_0|}\leq\log 2\leq u_D/2,$$

so that

$$\int_D \left|u(x) - u_D\right|^p \, dm \geq \int_B \left|u(x) - u_D\right|^p \, dm \geq \frac{(u_D)^p \omega_n r^n}{2^{n+p}}. \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$u_D \leq 4c\alpha\left(2^n m(D)\, d(x, \partial D)^{-n}/\omega_n\right)^{1/p}.$$

By Minkowski's inequality we also have

$$\begin{aligned}\left(⨍_D \left|u(x)\right|^p \, dm\right)^{1/p} &\leq \left(⨍_D \left|u(x) - u_D\right|^p \, dm\right)^{1/p} + \left(⨍_D u_D^p \, dm\right)^{1/p} \\ &\leq (2c\alpha) + 4c\alpha\left(2^n m(D)\, d(x, \partial D)^{-n}/\omega_n\right)^{1/p} = a. \quad \square\end{aligned}$$

2.6. Theorem. *Suppose D satisfies*

$$\left(⨍_D k(x, x_0)^p \, dm\right)^{1/p} \leq a \tag{2.7}$$

for some fixed point x_0 in D. Then D is an L^p-averaging domain and the constant c in (1.3) depends only on n, p and a.

Note that the choice of the fixed point x_0 is immaterial.

In proving Theorem 2.6 we use the following three lemmas and obtain the stronger conclusion, namely that if (2.7) holds then D satisfies (1.4) where c depends only on τ, n, p, and a.

2.8. Lemma. *If*

$$\sup_{\tau B \subset D} \left(\frac{1}{m(B)} \int_B |u - u_B|^p \, dm\right)^{1/p} \leq c_0, \tag{2.9}$$

then there exist positive constants s and $q = q'/c_0$ with s and q' depending only on n such that,

$$m\left\{x \in B : \left|u(x) - u_B\right| > t\right\} < s e^{-qt} m(B) \tag{2.10}$$

for all balls B satisfying $n\tau B \subset D$.

Lemma 2.8 follows from elementary geometry and the John–Nirenberg theorem [JN] and the observation that for $p \geq 1$,

$$\frac{1}{m(B)} \int_B |u - u_B| \, dm \leq \left(\frac{1}{m(B)} \int_B |u - u_B|^p \, dm\right)^{1/p}.$$

The key step in the proof of Theorem 2.6 involves comparing the average of u over various balls in D. Lemma 2.11 gives an estimate in terms of the quasihyperbolic metric alone.

2.11. Lemma. *Suppose that* s, q, τ *are constants with* $\tau > 1$ *and that*

$$m\left\{x \in B : \left|u - u_B\right| > t\right\} \le se^{-qt}m(B) \tag{2.12}$$

for each $t > 0$ *and each* B *satisfying* $\tau B \subset D$. *Then there exists a constant* $c = c(\tau, s, q, n)$ *such that*

$$\left|u_{B(x)} - u_{B(y)}\right| \le c\big(k(x,y) + 1\big), \tag{2.13}$$

for all x *and* y *in* D; *here* $B(x)$ *and* $B(y)$ *denote the balls* $B\big(x, d(x,\partial D)/\tau\big)$ *and* $B\big(y, d(y,\partial D)/\tau\big)$.

Proof. To simplify computations, assume $\tau \ge 3$; the same proof holds for $\tau \in (1,3)$ with minor changes. For each $z \in D$, let $B(z) = B(z,r)$, with $r = d/\tau$ where $d = d(z,\partial D)$, so that $\tau B(z) \subset D$. Fix $x, y \in D$ and choose a quasihyperbolic geodesic arc γ joining x to y in D. We use induction to define an ordered sequence of points $\{z_j\}$ on γ as follows. First set $z_1 = x$. Next suppose that $z_1, \ldots, z_j$ have been defined and let $\beta_j = \gamma(z_j, y)$ denote that part of γ from z_j to y and γ_j the component of $\beta_j \cap B(z_j)$ which contains z_j. Define z_{j+1} as the other endpoint of γ_j.

We simplify notation as follows: $B_j = B(z_j)$, $r_j = r(B_j)$, $d_j = d(z_j, \partial D)$ and $y = z_{m+1}$. From our definition of $\{z_j\}$, we see that

$$|z_{j+1} - z_j| = r_j \qquad \text{for} \quad j = 1 \text{ to } m-1, \qquad \text{and} \quad |z_{m+1} - z_m| \le r_m. \tag{2.14}$$

For $j = 1, \ldots, m-1$ pick $z_j' \in \partial D$ so that $d(z_j, \partial D) = |z_j - z_j'|$. If $z \in \gamma_j \subset B_j$, then

$$d(z,\partial D) \le |z - z_j'| \le |z - z_j| + |z_j - z_j'| \le 4d_j/3.$$

Hence

$$\int_{\gamma_j} \frac{1}{d(z,\partial D)}\, ds \ge \frac{3}{4d_j} \int_{\gamma_j} ds \ge \frac{3|z_{j+1} - z_j|}{4d_j} = \frac{3}{4\tau}.$$

Summing the above over j gives

$$\frac{3(m-1)}{4\tau} \le \sum_1^{m-1} \int_{\gamma_j} \frac{1}{d(z,\partial D)}\, ds \le \int_{\gamma} \frac{1}{d(z,\partial D)}\, ds = k(x,y), \tag{2.15}$$

and we conclude $m < \infty$.

Consider now the relative size of neighboring balls. Fix j and choose z, $z' \in \partial D$ so that $d_j = |z - z_j|$ and $d_{j+1} = |z' - z_{j+1}|$. Then $d_j \le d_{j+1} + r_j$ and $d_{j+1} \le d_j + r_j$, with the first inequality yielding

$$r_{j+1} = d_{j+1}/\tau \ge (d_j - r_j)/\tau \ge 2r_j/3,$$

and the second yielding $r_j \geq 2r_{j+1}/3$ if $r_{j+1} \geq r_j$. Hence

$$2r_j/3 \leq r_{j+1} \leq 3r_j/2. \tag{2.16}$$

Next we show that

$$m(B_j \cap B_{j+1}) > c_1\big(m(B_j) + m(B_{j+1})\big), \tag{2.17}$$

where $c_1 = 6^{-n}/2$. Fix j and set $w = (z_j + z_{j+1})/2$, $s = \big(\max(r_j, r_{j+1})\big)/6$ and $B = B(w,s)$. In the case $r_j \geq r_{j+1}$ we have $s = r_j/6 \leq r_{j+1}/4$; similarly if $r_{j+1} \geq r_j$ we get $s \leq r_j/4$. Thus for $z \in B$ we have,

$$|z - z_j| \leq |z - w| + |w - z_j| \leq s + \tfrac{1}{2}|z_j - z_{j+1}| \leq r_j;$$

a similar argument shows $|z - z_{j+1}| \leq r_{j+1}$. Hence we conclude $B \subset B_j \cap B_{j+1}$ and (2.17) follows.

Now for $j = 1, 2, \ldots, m+1$ let

$$E_j = \big\{x \in B_j : \big|u(x) - u_{B_j}\big| > t\big\},$$

where $t = (\log 2s/c_1)/q$. By (2.12),

$$m(E_j) \leq c_1 m(B_j)/2 \tag{2.18}$$

and combining (2.17) and (2.18) yields

$$m\big((B_j \cap B_{j+1}) \setminus (E_j \cup E_{j+1})\big) > 0.$$

Therefore there exists $x \in (B_j \cap B_{j+1}) \setminus (E_j \cup E_{j+1})$ and hence

$$|u_{B_j} - u_{B_{j+1}}| \leq |u(x) - u_{B_j}| + |u(x) - u_{B_{j+1}}| \leq 2t.$$

Summing and using (2.15) we obtain

$$\big|u_{B(x)} - u_{B(y)}\big| \leq \sum_{1}^{m} |u_{B_j} - u_{B_{j+1}}| \leq 2mt \leq 8\tau\big(k(x,y)\big)t/3 + 2t.$$

Then (2.13) follows with $c = 8\tau\log(4s6^n)/3q$. □

2.19. Lemma. *Suppose s, q, and τ are constants with $\tau > 1$ and that*

$$m\{x \in B : |u(x) - u_B| > t\} \leq s e^{-qt} m(B) \tag{2.20}$$

for each $t > 0$ and each ball B satisfying $\tau B \subset D$. If D satisfies (2.7), then

$$\left(\frac{1}{m(D)} \int_D |u - u_D|^p \, dm\right)^{1/p} \leq b, \tag{2.21}$$

where $b = b(s, q, \tau, n, p, a)$.

Proof. Let $\lambda(t) = m\{x \in D : k(x, x_0) > t\}$. Then by (2.7),

$$\int_0^\infty pt^{p-1}\lambda(t)\,dt = \int_D k(x, x_0)^p\,dm \le a^p m(D).$$

Let c be the constant in (2.13), fix $t > 0$ and let

$$E_t = \{x \in D : |u(x) - u_{B(x_0)}| > t\},$$

$$F = \{x \in D : k(x, x_0) \le t_2\}.$$

Here t_2 is a function of t to be chosen later. We estimate $m(E_t)$ by first noting that

$$m(E_t \setminus F) \le m(D \setminus F) = \lambda(t_2). \tag{2.22}$$

By Lemma 2.11, for each $x \in F$ we get a closed ball $B(x)$ with $\tau B(x) \subset D$ such that

$$\left|u_{B(x_0)} - u_{B(x)}\right| \le c\big(k(x, x_0) + 1\big) \le c(t_2 + 1).$$

The set F is bounded since for any points $x, y \in F$ we have $k(x, y) \le 2t_2 < \infty$ [GP]. Here for each $x \in F$,

$$d(x, \partial D) \le \operatorname{diam} F + d(x_0, \partial D) = d,$$

so that the radii of the balls $B(x)$ are uniformly bounded. The union of all such balls covers F and we can apply a well known covering theorem from page 9 of [S] to obtain a subcover of balls $B_j = B(x_j)$, $x_j \in F$ with the following properties:

(i) $F \subset \bigcup_j B_j$,
(ii) The balls $B'_j = B_j/5$ are pairwise disjoint.

This gives the useful relation:

$$\sum_j m(B_j) = 5^n \sum_j m(B'_j) \le 5^n m(D). \tag{2.23}$$

Now fix $x \in E_t \cap B_j$. Then since $x_j, x_0 \in F$,

$$|u(x) - u_{B(x_j)}| \ge |u(x) - u_{B(x_0)}| - |u_{B(x_j)} - u_{B(x_0)}| \ge t - c(t_2 + 1) = t_1,$$

and hence

$$m(E_t \cap F) \le \sum_j m(E_t \cap B_j) \le 5^n s e^{-q t_1} m(D), \tag{2.24}$$

by (2.20) and (2.23). Combining (2.22) and (2.24) then yields

$$m(E_t) \le 5^n s e^{-qt_1} m(D) + \lambda(t_2).$$

Let $t_2 = (t/2c)$ so that $t_1 = (t/2) - c$, and then

$$m(E_t) \le 5^n s e^{qc} e^{-qt/2} m(D) + \lambda(t/2c).$$

As

$$\left(\frac{1}{m(D)} \int_D \left|u(x) - u_{B(x_0)}\right|^p dm\right)^{1/p} = \left(\frac{1}{m(D)} \int_0^\infty p t^{p-1} m(E_t)\, dt\right)^{1/p}$$

we have

$$\left(\frac{1}{m(D)} \int_D \left|u(x) - u_{B(x_0)}\right|^p dm\right)^{1/p} \le (c_1 q^{-p} + c_2 a^p c^p)^{1/p}, \tag{2.25}$$

where c_1 and c_2 depend only on n, s, τ, and p. Now (2.21) follows with $b = 2(c_1 q^{-p} + c_2 a^p c^p)^{1/p}$. □

We prove Theorem 2.6 by recalling the definitions of the appropriate constants in each of the three lemmas. Only $q = q'/c_0$ and the constant c in (2.13) depend on c_0 in (2.9) and (2.21) can be rewritten to give

$$\left(\frac{1}{m(D)} \int_D |u - u_D|^p\, dm\right)^{1/p} \le c' c_0,$$

where $c' = c'(n, \tau, a, p)$.

2.26. Corollary. *Let $\tau > 1$ be a constant. If*

$$\sup_{\tau B \subset \Omega} \frac{1}{m(B)} \int_B |u - u_B|\, dm \le c_0$$

for some constant c_0, then $u \in \mathrm{BMO}(\Omega)$, with $\|u\|_* \le c c_0$, $c = c(n, \tau)$.

Proof. From the calculations in Theorem 2.1 we see balls satisfy (2.7) for $p = 1$. Theorem 2.6 then gives

$$\sup_{B \subset \Omega} \frac{1}{m(B)} \int_B |u - u_B|\, dm \le c \sup_{\tau B \subset \Omega} \frac{1}{m(B)} \int_B |u - u_B|\, dm. \ \square$$

3. The Poincaré inequality and examples of L^p-averaging domains

We begin this section by observing that L^p-averaging domains are preserved under quasi-isometries. This result will aid our computations in proving the Poincaré inequality theorem and other theorems throughout Section 3.

Definition. A mapping f defined in D is said to be a K*-quasi-isometry*, $K > 1$, if

$$\frac{1}{K} \le \frac{|f(x) - f(y)|}{|x - y|} \le K,$$

for all $x, y \in D$.

3.1. Theorem. *If $f\colon D \to D'$ is a quasi-isometry and D is an L^p-averaging domain, then D' is an L^p-averaging domain.*

Proof. If f is a K-quasi-isometry, then $1/K^n \le J(f) \le K^n$ a.e., where $J(f)$ is the Jacobian of f. Let γ be a quasihyperbolic geodesic joining x to y in D and set $\gamma' = f(\gamma)$. We can easily check that

$$k\big(f(x), f(y); D'\big) \le \int_{\gamma'} \frac{ds'}{d\big(f(z), \partial D'\big)} \le \int_{\gamma} \frac{K^2 ds}{d(z, \partial D)} = K^2 k(x, y; D).$$

Combining these observations yields

$$\left(⨍_{D'} k\big(f(x), f(x_0); D'\big)^p \, dm \right)^{1/p} \le a(K^{2n+2p})^{1/p},$$

where a is as in (2.7). □

Using Theorem 3.1 we can easily estimate the integral of $k(x, x_Q)$ over a cube Q with center x_Q. This result will be needed when we cover the domain D with cubes, such as in the Whitney cube decomposition of D.

3.2. Corollary. *For any cube Q with center x_Q, we have*

$$\left(⨍_{Q} k(x, x_Q)^p \, dm \right)^{1/p} \le c, \qquad c = c(n, p). \tag{3.3}$$

Proof. For any cube Q there exists a quasi-isometry f mapping Q to $B(x_Q, l(Q)/2)$ with $f(x_Q) = x_Q$. By the calculations done for balls in Theorem 2.1 and by Theorem 3.1 we have (3.3).

L^p-averaging domains constitute a large class of domains in $\mathbf{R}^n$ which satisfy a Poincaré type inequality [H], [M].

3.4. Theorem. *If D is an L^p-averaging domain, $p \geq n$, then there exists a constant c, such that*

$$(3.5) \qquad \left(\frac{1}{m(D)}\int_D |u-u_D|^p\,dm\right)^{1/p} \leq cm(D)^{1/n}\left(\frac{1}{m(D)}\int_D |\nabla u|^p\,dm\right)^{1/p},$$

for each function u in the Sobolev class $W_1^p(D)$.

Proof. For each ball $B \subset D$, we see from (7.45) in [GT] that

$$\begin{aligned}\left(\frac{1}{m(B)}\int_B |u-u_B|^p\,dm\right)^{1/p} &\leq c_1 m(B)^{(1/n)-(1/p)}\left(\int_B |\nabla u|^p\,dm\right)^{1/p}\\ &\leq c_1 m(D)^{1/n}\left(\frac{1}{m(D)}\int_D |\nabla u|^p\,dm\right)^{1/p}.\end{aligned}$$

Thus (3.5) follows from (1.3). □

The following shows that the hypothesis $p \geq n$ is essential in Theorem 3.4.

3.6. Theorem. *If $p < n$, there exists a domain $D \subset \mathbf{R}^n$ which is L^q-averaging for all $q < \infty$, such that* (3.5) *does not hold for any constant c.*

Proof. Construct D by alternately adjoining large cubes, $\{Q_i\}$, with small cubes, $\{R_i\}$, with centers x_i and y_i respectively, so that

(i) Q_0 is centered at the origin,
(ii) each cube is centered on the positive x_1 axis, and
(iii) the x_1 coordinates of the centers form an increasing sequence when the centers are ordered as follows:

$$\{x_0, y_1, x_1, y_2, x_2, \ldots, x_i, y_{i+1}\}.$$

Next join this domain to its reflection in $x_1 = 0$ to get a symmetric domain. The sidelengths of the cubes in D are

$$l(Q_i) = l(Q_{-i}) = q_i = 2^{-(i+1)} \qquad \text{for} \quad i = 0, 1, 2, \ldots$$

and

$$l(R_i) = l(R_{-i}) = r_i = 2^{-ai} \qquad \text{for} \quad i = 1, 2, \ldots \quad \text{with } a > 1,$$

where Q_{-i} and R_{-i} are the reflections of Q_i and R_i respectively.

We show first that D is an L^q-averaging domain for each finite q. Decomposing D gives

$$(3.7) \qquad \int_D k(z,0)^q\,dm \leq 2\sum_1^\infty\left(\int_{Q_{i-1}} k(z,0)^q\,dm + \int_{R_i} k(z,0)^q\,dm\right).$$

By (3.3) we have

$$\text{(3.8)}\quad \int_{Q_i} k(z,0)^q\,dm \le \int_{Q_i} \big(k(z,x_i)+k(x_i,0)\big)^q\,dm \le 2^q\big(c_1+k(x_i,0)^q\big)m(Q_i).$$

We get a similar statement for each R_i with center y_i. Now estimating the quasihyperbolic metric geometrically we get

$$k(x_{i-1},0) \le k(y_i,0) \le 2\left(\sum_0^i 2+\log\frac{q_j}{r_{j+1}}\right) = c_2 i + c_3 i^2.$$

Thus

$$\text{(3.9)}\qquad k(x_{i-1},0)^q \le k(y_i,0)^q \le c_4 i^{2q}.$$

Substituting (3.8) and (3.9) in (3.7) yields

$$\text{(3.10)}\qquad \int_D k(z,0)^q\,dm \le 2\Big(\sum_1^\infty 2^q\big(c_1+c_4 i^{2q}\big)\big(2^{-ni}+2^{-nai}\big)\Big) < \infty,$$

so that D is an L^q-averaging domain for each $q<\infty$.

Now we construct a function u so that (3.5) does not hold in D. Define u to be the piecewise linear function in x_1 which satisfies $u=0$ on Q_0 and

$$\frac{du}{dx_1} = \begin{cases} 2^{b|i|} & \text{on } R_i \text{ for all } i, \\ 0 & \text{on } Q_i \text{ for all } i \end{cases}$$

so that $u_D=0$. Here b is chosen so that

$$\text{(3.11)}\qquad \frac{n+pa}{p} < b < \frac{na}{p}.$$

This is possible if and only if we choose a so that $a > n/(n-p)$.

Now let us estimate each of the integrals in (3.5). On the right hand side we have,

$$\text{(3.12)}\qquad \int_D |\nabla u|^p\,dm = 2\sum_1^\infty 2^{(bp-an)i} < \infty,$$

by (3.11).

We get a lower bound on the left hand side by noting,

$$|u| \ge 2^{bi-ai} \qquad \text{on } Q_i \text{ and } Q_{-i} \quad \text{for} \quad i>1.$$

Thus

$$\int_D |u - u_D|^p \, dm \geq 2 \sum_1^{\infty} 2^{(bp-ap-n)i} = \infty, \tag{3.13}$$

by (3.11). Therefore (3.5) does not hold in the case $p < n$. □

John domains, studied by Martio and Sarvas [MS] in their work on injectivity, are a subclass of L^p-averaging domains. Many well known domains are John domains including bounded convex domains, bounded quasidisks and bounded uniform domains. Martio has recently shown that John domains satisfy a Poincaré inequality for exponents $p \geq 1$ [M].

Definition. Let $0 < \alpha \leq \beta < \infty$. A domain $D \subset \mathbf{R}^n$ is called an (α, β) *John domain*, if there is an $x_0 \in D$ such that every $x \in D$ has a rectifiable path γ: $[0, d] \to D$ with arclength as parameter such that $\gamma(0) = x$, $\gamma(d) = x_0$, $d \leq \beta$ and

$$d\big(\gamma(t), \partial D\big) \geq \frac{\alpha}{d} t \qquad \text{for all} \quad t \in [0, d].$$

We prove that John domains are L^p-averaging domains.

3.14. Theorem. *If D is a John domain, then D is an L^p-averaging domain.*

Proof. By a result of Boman [B], there exist constants $\tau > 1$, $N \geq 1$, and a covering Γ for D consisting of open cubes Q and a distinguished cube Q_0 with the following properties:

(3.15) $\sum_{Q \in \Gamma} \chi_{\tau Q}(x) \leq N \chi_D(x)$, $x \in \mathbf{R}^n$, and

(3.16) Each cube Q in Γ can be connected to Q_0 by a chain of cubes Q_0, $Q_1, \ldots,$ $Q_s = Q$ from Γ such that for each $j = 0, 1, \ldots, s-1$, $Q \subset N Q_j$, and there exists a cube $R_j \subset Q_j \cap Q_{j+1}$ which satisfies $Q_j \cup Q_{j+1} \subset N R_j$.

We proceed using integration techniques found in [IN] and the notation

$$k_Q = \fint_Q k(x, x_0) \, dm.$$

Let x_0 be the center of Q_0. From (3.3) we deduce

(3.17)
$$\left(\fint_Q \left| k(x, x_0) - k_Q \right|^p dm \right)^{1/p} \leq 2 \left(\fint_Q \left| k(x, x_0) - k(x_0, x_Q) \right|^p dm \right)^{1/p} \leq 2c.$$

Now using the cover Γ and the elementary inequality

$$|a + b|^p \leq 2^p \big(|a|^p + |b|^p \big),$$

we have

$$\int_D |k(x,x_0) - k_{Q_0}|^p \, dm \leq 2^p \Big(\sum_{Q\in\Gamma} \int_Q |k(x,x_0) - k_Q|^p \, dm + \sum_{Q\in\Gamma} \int_Q |k_Q - k_{Q_0}|^p \, dm \Big). \tag{3.18}$$

Estimating the first term using (3.15) and (3.17) gives,

$$\sum_{Q\in\Gamma} \int_Q |k(x,x_0) - k_Q|^p \, dm \leq \sum_{Q\in\Gamma} (2c)^p m(Q) \leq (2c)^p N m(D).$$

For the second term, fix a cube $Q = Q_s$ and let Q_0, $Q_1, \dots,$ Q_s be the chain given in (3.16). Since $Q_j \cup Q_{j+1} \subset NR_j$,

$$⨍_{R_j} |k(x,x_0) - k_{Q_j}|^p \, dm \leq \frac{m(Q_j)}{m(R_j)} ⨍_{Q_j} |k(x,x_0) - k_{Q_j}|^p \, dm \leq (2c)^p N^n,$$

and thus

$$|k_{Q_j} - k_{Q_{j+1}}|^p \leq 2(4c)^p N^n.$$

Property (3.16) gives

$$|k_{Q_j} - k_{Q_{j+1}}|^p \chi_Q(x) \leq 2(4c)^p N^n \chi_{NQ_j}(x),$$

so that

$$\begin{aligned} |k_{Q_0} - k_Q|^p \chi_Q(x) &\leq \Big(\sum_0^{s-1} |k_{Q_j} - k_{Q_{j+1}}| \chi_Q(x) \Big)^p \\ &\leq 2(4c)^p N^n \Big(\sum_{R\in\Gamma} \chi_{NR}(x) \Big)^p. \end{aligned}$$

Now any point x is in at most N cubes Q in Γ by (3.15); thus

$$\int_{\mathbf{R}^n} \sum_{Q\in\Gamma} |k_{Q_0} - k_Q|^p \chi_Q(x) \, dm \leq 2(4c)^p N^{n+1} \int_{\mathbf{R}^n} \Big(\sum_{R\in\Gamma} \chi_{NR}(x) \Big)^p \, dm.$$

By Lemma 4 of [IN],

$$\int_{\mathbf{R}^n} \Big(\sum_{R\in\Gamma} \chi_{NR}(x) \Big)^p \, dm \leq c_1 \int_{\mathbf{R}^n} \Big(\sum_{R\in\Gamma} \chi_R(x) \Big)^p \, dm,$$

with $c_1 = c_1(N,n,p)$ so that

$$\sum_{Q\in\Gamma}\int_Q |k_{Q_0} - k_Q|^p\,dm \le 2c_1(4c)^p N^{n+1+p} m(D).$$

Substituting these two estimates in (3.18) and using (3.3), we obtain,

$$\left(⨍_D k(x,x_0)^p\,dm\right)^{1/p} \le a, \qquad a = a(N,n,p). \ \square$$

John domains may have internal cusps, but no external cusps. Certain outwardly directed spires are allowed in L^p-averaging domains and the following theorem characterizes such finite spires.

3.19. Theorem. *Let D be the domain $Q\cup S\subset \mathbf{R}^n$, where Q is the cube*

$$Q = \{(x_1,x_2,\ldots,x_n) : |x_1-2|, |x_2|, \ldots, |x_n| < 1\},$$

and S is the spire

$$S = \left\{(x_1,x_2,\ldots,x_n) : \sum_2^n (x_i)^2 < g(x_1)^2, \quad 0\le x_1 < 1\right\} \tag{3.20}$$

and where $g(x)$ satisfies the following properties:
(i) $g(0)=0$, $g(1)\le 1$,
(ii) $0 < g'(x) \le M$, *for* $0 < x \le 1$,
(iii) $g''(x)\ge 0$, *for* $0\le x\le 1$.
Then D is an L^p-averaging domain if and only if

$$\int_0^1 g(x)^{n-1}\left(\int_x^1 \frac{1}{g(t)}\,dt\right)^p dx < \infty. \tag{3.21}$$

Proof. We take $z_0 = (1,0,0,\ldots,0)$ for our fixed point and estimate $k(z,z_0)$ for $z = (a_1,a_2,\ldots,a_n)\in S$ as follows. We let $z_1 = (a_1,0,\ldots,0)$ and obtain an upper bound on $k(z,z_1)$ by examining the cross section of S where $x_1 = a_1$. This gives

$$k(z,z_1)\le \log\frac{R}{R-r}, \quad \text{where} \quad R = g(a_1) \quad \text{and} \quad r^2 = \sum_2^n (a_i)^2. \tag{3.22}$$

For any point $y = (x_1,0,\ldots,0)$, note that the distance to the boundary satisfies

$$g(x_1)\ge d(y,\partial D)\ge g(x_1)\cos\theta, \qquad \text{where} \quad \tan\theta = g'(x_1).$$

Thus

$$\frac{1}{g(x_1)} \le \frac{1}{d(y,\partial D)} \le \frac{1}{g(x_1)}\left(g'(x_1)^2+1\right)^{1/2} \le \frac{c}{g(x_1)}, \tag{3.23}$$

by (ii).

Combining (3.22) and (3.23) yields

$$\left(\int_{a_1}^1 \frac{1}{g(t)}\,dt\right)^p \le k(z,z_0)^p \le 2^p\left(\left(\log\frac{g(a_1)}{g(a_1)-r}\right)^p + \left(c\int_{a_1}^1 \frac{1}{g(t)}\,dt\right)^p\right).$$

By a change of coordinates, we have

$$\int_S k(z,z_0)^p\,dm \le c_1\int_0^1\int_0^{g(x)}\left(\log\frac{g(x)}{g(x)-r}\right)^p r^{n-2}\,dr\,dx + c_2\int_0^1\int_0^{g(x)}\left(\int_x^1 \frac{1}{g(t)}\,dt\right)^p r^{n-2}\,dr\,dx.$$

The first integral is finite for all $n \ge 2$ and $1 \le p < \infty$, so we need only consider the behavior of the second integral. After integration with respect to r this reduces to (3.21). By (3.3) we can get a bound on the integral of $k(z,z_0)^p$ over Q, thus completing the proof. □

Similarly we can prove that an infinite spire of revolution S in $\mathbf{R}^n$ given by

$$S = \left\{(x_1,x_2,\ldots,x_n) : \sum_2^n (x_i)^2 < g(x_1)^2, \quad 0 \le x_1 < \infty\right\}$$

is an L^p-averaging domain if and only if

$$\int_0^\infty g(x)^{n-1}\left(\int_0^x \frac{1}{g(t)}\,dt\right)^p < \infty. \tag{3.24}$$

The above two results can be applied to get specific examples. For instance, finite spires S of revolution generated by $g(x_1) = x_1^\alpha$, for $\alpha > 1$ and $p+1 > n$, are L^p-averaging domains if and only if

$$\alpha < \frac{p+1}{p+1-n}. \tag{3.25}$$

Similarly the infinite spire S generated by $g(x_1) = x_1^\alpha$, for $p+1 < n$ is an L^p-averaging domain if and only if (3.25) holds.

These examples serve to differentiate L^p-averaging domains from John domains in another way. Using these domains we can show that the class of L^p-averaging domains is not preserved under quasiconformal maps.

3.26. Theorem. *L^p-averaging domains are not invariant with respect to quasiconformal self mappings of $\mathbf{R}^n$.*

Proof. We use spires of revolution S in each of the following three cases and the quasiconformal mapping, $f(z) = |z|^{k-1}z$. Denote $f(S)$ by S'.

Case i: $p+1 > n$. Here consider the finite spire S generated by

$$g(x_1) = x_1^\alpha, \qquad \text{with} \quad \alpha > \frac{p+1}{p+1-n},$$

so that S is not an L^p-averaging domain. The spire S' is an L^p-averaging domain for large values of K.

Case ii: $p+1 < n$. We take the infinite spire S generated by

$$g(x_1) = x_1^\alpha, \qquad \text{with} \quad \alpha < \frac{p+1}{p+1-n},$$

so that S is an L^p-averaging domain. Now S' is not an L^p-averaging domain for large values of K.

Case iii: $p+1 = n$. In this case consider the infinite spire S generated by

$$g(x_1) = x_1^{1-K} e^{-x_1{}^K}, \qquad \text{with} \quad K > \frac{n}{n-1}$$

so that by (3.24) S is an L^p-averaging domain. By means of (3.24) again, we see that S' is not an L^p-averaging domain. □

Also in contrast to the situation for John domains, the boundary of an L^p-averaging domain can have positive n-dimensional measure. Specifically, for $p \leq n-1$, we can construct an L^p-averaging domain satisfying $m(\partial D) = \infty$. Since the class of L^p-averaging domains is a decreasing set with respect to p, an example for the case $p = n-1$ will suffice. We give the proof here for the case $n = 2$ and $p = 1$ and sketch the idea for the generalization to $p = n-1$, $n \geq 2$.

3.27. Example. We construct an L^1-averaging domain D in $\mathbf{R}^2$ with $m(\partial D) = \infty$. Let Q be the unit square centered at $z_0 = (-1/2, 0)$ and let F be the edge $x = 0$.

On F we choose a countable dense set of points as follows. Let F_i denote the dyadic decomposition of F into 2^i intervals and define C_i as the set of all centers of F_i. The set of all centers,

$$C = \bigcup_1^\infty C_i$$

is dense in F and the number of points in each C_i is

$$|C_i| = 2^i.$$

Now for each point z_{ij} in C_i, we take the closed interval B_{ij} of length $r_i = 2^{-2i-1}$ centered at z_{ij}. The lengths are thus chosen so that

$$m(B_{ij})|C_i| = 2^{-i-1}.$$

We define the set E as

$$E = F \setminus \bigcup_i \Big(\bigcup_{z_{ij} \in C_i} B_{ij} \Big).$$

(Note that for all future unions over both i and j as above the notation will be abbreviated $\cup_{i,j}$.) We have

$$m(E) \geq m(F) - \sum_i 2^{-i-1} = 1/2.$$

We attach the spire S_{ij} to base B_{ij} where S_{ij} is a suitable translation of the spire S_i generated by

$$g(x) = e^{-tx}(2t)^{-e^{tx}}, \qquad \text{with} \quad t = (r_i)^{-1}.$$

Finally define D as

$$D = Q \cup \bigcup_{i,j} S_{ij}.$$

The boundary of D contains $[0, \infty) \times E$, so that $m(\partial D) = \infty$.

Now we show that D is an L^p-averaging domain. In the following computations, c denotes any positive constant depending only on the dimension n.

First by using the technique in Theorem 3.19 we obtain

$$\int_{S_i} k(z,0)\, dm \leq M \int_0^\infty \int_0^{g(x)} \left(\log \frac{g(x)}{g(x) - r} + \int_0^x \frac{1}{g(s)}\, ds \right) dr\, dx,$$

where

$$M = \max_{x \geq 0} \big(g'(x)^2 + 1\big)^{1/2} \leq 2 + \log 2t.$$

After performing the integration the above simplifies to

$$\int_{S_i} k(z,0)\, dm \leq ct^{-2}. \tag{3.28}$$

We have the following inequality,

(3.29)

$$\int_D k(z_0,0)\, dm < \sum_{i,j} \left(\int_{S_{ij}} k(z_0, z_{ij})\, dm + \int_{S_{ij}} k(z_{ij}, z)\, dm \right) + \int_Q k(z, z_0)\, dm.$$

We can obtain the following estimates for $k(z_0, z_{ij})$ and $m(S_{ij})$,

$$k(z_0, z_{ij}) \leq 1 - c \log r_i \qquad \text{and} \quad m(S_{ij}) \leq ct^{-2}.$$

Substituting these and the value of r_i into 3.29, and using (3.3), we see that the sum in (3.29) converges or that D is an L^1-averaging domain.

To generalize to the case $p = n-1$ for all $n \geq 2$, we take the countable dense set of centers generated by the dyadic decomposition of the face $x_1 = 0$ in the n-dimensional cube Q. Define balls B_{ij} of radius

$$r_i = 2^{(-ni-1)/(n-1)}(\omega_{n-1})^{-1/(n-1)}$$

and attach spires of revolution generated by the function $g(x_1)$. The desired conclusions follow after similar but more involved computations. □

The classical case of the Poincaré inequality, with $p = n = 2$, has been studied more extensively. For this case we can get the following stronger version of Theorem 3.4.

3.30. Theorem. *Let D be a simply connected domain in $\mathbf{R}^2$. If D is an L^1-averaging domain, then there is a constant c, such that*

$$(3.31) \qquad \left(\frac{1}{m(D)} \int_D |u - u_D|^2 \, dm\right)^{1/2} \leq cm(D)^{1/2} \left(\frac{1}{m(D)} \int_D |\nabla u|^2 \, dm\right)^{1/2},$$

for each function $u \in L^2(D) \cap W_1^2(D)$.

Proof. Hamilton [H] has shown that if there exists a point $z_0 \in D$ and constant c such that

$$(3.32) \qquad \int_D |f|^2 \, dm \leq c \int_D |f'|^2 dm,$$

for all f analytic in D normalized so that $f(z_0) = 0$, then D satisfies (3.31).

Also note that the quasihyperbolic metric, $k(x, y; D)$, and the hyperbolic metric, $h(x, y; D)$ are equivalent metrics in simply connected domains D in $\mathbf{R}^2$ [G].

With the above two results the proof of Theorem 3.30 can now be reduced to showing that for some $z_0 \in D$ and for all f analytic in D we have

$$(3.33) \qquad \int_D \left|f(z) - f(z_0)\right|^2 dm \leq \frac{1}{\pi} \left(\int_D |f'|^2 \, dm\right) \int_D h(z, z_0; D) \, dm.$$

We show (3.33) as follows. Let B denote the unit disk, choose $\varphi \colon B \to D$ conformal with $\varphi(0) = z_0$, and set $g = f \circ \varphi$.

Next suppose that

$$g(w) = \sum_0^\infty a_n w^n.$$

Then

$$|f(z) - f(z_0)|^2 = |g(w) - g(0)|^2 = \left|\sum_1^\infty a_n w^n\right|^2.$$

By Hölder's inequality we have,

$$\begin{aligned}
|f(z) - f(z_0)|^2 &\le \left(\sum_1^\infty n|a_n|^2\right)\left(\sum_1^\infty \frac{|w|^{2n}}{n}\right) \\
&= \left(\frac{1}{\pi}\int_B |g'(w)|^2\, dm\right)\left(\log\frac{1}{1-|w|^2}\right) \\
&\le \left(\frac{1}{\pi}\int_D |f'|^2\, dm\right)\log\frac{1+|w|}{1-|w|} \\
&= \left(\frac{1}{\pi}\int_D |f'|^2\, dm\right)h(w,0;B) = \left(\frac{1}{\pi}\int_D |f'|^2\, dm\right)h(z,z_0;D).
\end{aligned}$$

This pointwise estimate clearly gives (3.33). □

Combining Theorem 3.30 with Example 3.27 gives us a domain D with infinite boundary measure that satisfies the Poincaré inequality. Theorem 3.30 also clearly shows that the hypothesis that D is an L^p-averaging domain in Theorem 3.4 is not necessary.

Finally we observe that L^p-averaging domains can be characterized in terms of their covers by Whitney cubes [S].

3.34. Theorem. *If the Whitney cube cover F of D consists of cubes Q_j with centers x_j, then the following two conditions are equivalent:*

(3.35)
$$\left(\frac{1}{m(D)}\int_D k(x,x_0)^p\, dm\right)^{1/p} < \infty,$$

(3.36)
$$\left(\frac{1}{m(D)}\sum_{Q_j\in F} k(x_j,x_0)^p m(Q_j)\right)^{1/p} < \infty.$$

Proof. Assume (3.35) holds. Then by (3.3),

$$\left(\int_D k(x,x_0)^p\,dm\right)^{1/p}$$

$$\leq 2^{1/p}\left(\left(\sum_{Q_j}\int_{Q_j} k(x_j,x)^p\right)^{1/p}+\left(\sum_{Q_j}\int_{Q_j} k(x_j,x_0)^p\right)^{1/p}\right)$$

$$\leq \big(2m(D)\big)^{1/p}c_n+2^{1/p}\left(\sum_{Q_j} k(x_j,x_0)^p m(Q_j)\right)^{1/p}.$$

Thus (3.36) follows clearly. The implication in the other direction is proven similarly. □

3.37. Corollary. *If D is an L^p-averaging domain and if N_j denotes the number of cubes of side length 2^{-j} in the Whitney decomposition F of D, then*

$$\sum_j j^p 2^{-nj} N_j < \infty. \tag{3.38}$$

Proof. From Gehring–Palka [GP] we have the inequality

$$k(x_0,x)\geq \log\frac{d(x_0,\partial D)}{d(x,\partial D)}.$$

Let Q_0 with center x_0 be the largest cube in F, and let Q_i with center x_i be of side length 2^{-j}. From the Whitney cube cover properties,

$$k(x_0,x_i)\geq \log\frac{d(x_0,\partial D)n^{-1/2}}{5l(Q_i)}=c+j\log 2.$$

Substituting this estimate in (3.36) we deduce (3.38). □

3.39. Remark. An example exists proving condition (3.38) is not sufficient.

4. Oscillation domains

We now turn our attention to the class of oscillation domains, i.e. the analogues of L^p-averaging domains with the average oscillation,

$$\left(\frac{1}{m(D)}\int_D |u-u_D|^p\,dm\right)^{1/p}$$

replaced by the maximal oscillation,

$$\operatorname*{osc}_D u=\sup_D u-\inf_D u.$$

Definition. We say that D is an *oscillation domain* if D satisfies

$$\operatorname*{osc}_D u\leq c\sup_{B\subset D}\big(\operatorname*{osc}_B u\big). \tag{1.8}$$

These domains can be characterized by a special chaining property.

4.1. Theorem. *A bounded domain $D \subset \mathbf{R}^n$ satisfies* (1.8) *if and only if for some constant N each pair of points x and y in D can be joined by a chain of balls B_0, B_1, ..., B_s in D such that $x \in B_0$, $y \in B_s$, $B_j \cap B_{j+1} \neq \emptyset$ and $s \leq N$.*

Proof. Assume first that D does not have this chaining property. We assign each ball B in D to an open set L_j as follows. Let B_0 be the largest ball contained in D and set $L_0 = B_0$. Next suppose that L_0, L_1, ..., L_{j-1} have been defined for some $j \geq 1$ and let L_j be the union of all balls B contained in D which satisfy:

(i) $B \cap L_{j-1} \neq \emptyset$,

(ii) $B \setminus \bigcup_0^{j-1} L_i \neq \emptyset$.

Clearly $D = \bigcup_0^\infty L_j$.

Next define u so that

$$u = \begin{cases} 0 & \text{on } L_0, \\ j & \text{on } L_j \setminus \bigcup_0^{j-1} L_i, \text{ for } j = 1, 2, \ldots \end{cases}$$

Since D does not have the chaining property, $\operatorname{osc}_D u = \infty$.

To show $\sup_{B \subset D}(\operatorname{osc}_B u)$ is bounded, let B be any ball in D and set

$$k = \inf\{j : (L_j \cap B) \neq \emptyset\}.$$

Then

$$B \subset (L_k \cup L_{k+1}) \setminus \bigcup_0^{k-1} L_j,$$

and hence u assumes at most the two values k and $k+1$ in B. Thus

$$\operatorname*{osc}_B u = \sup_B u - \inf_B u \leq 1.$$

For the sufficiency, suppose that $\sup_{B \subset D} \operatorname{osc} u \leq m$. Next for any two points x and y in D, let B_0, B_1, ..., B_s be the chain of balls connecting them and choose x_j in $B_{j-1} \cap B_j$. Then

$$\begin{aligned} \left|u(x) - u(y)\right| &\leq \left|u(x) - u(x_1)\right| + \sum_1^{s-1} \left|u(x_j) - u(x_{j+1})\right| + \left|u(x_s) - u(y)\right| \\ &\leq (s+1)m \leq (N+1)m. \end{aligned}$$

We have

$$\operatorname*{osc}_D u \leq (N+1) \sup_{B \subset D} (\operatorname*{osc}_B u). \quad \square \tag{4.2}$$

We conclude this paper by giving two geometric conditions, one necessary and one sufficient, for D to be an oscillation domain.

4.3. Theorem. *If D is an oscillation domain, then each point in the boundary of D is contained in the closure of a ball $B \subset D$.*

Proof. We proceed by induction on N, where N is the chaining constant given in Theorem 4.1. For the case $N = 0$, D itself is a ball and each $z \in \partial D$ is in the closed ball $\overline{D}$.

Assume that the result has been established for $N = k$. Consider all balls B in D and assign each to the open set L_j defined in the proof of Theorem 4.1. Let D be a domain with chaining constant $k+1$. Each ball $B \subset D$ is then in some L_j for $j \le k+1$. We write

$$D = \bigcup_1^{k+1} L_j = D_k \cup L_{k+1}, \qquad \text{where} \quad D_k = \bigcup_1^{k} L_j.$$

Choose $z_0 \in \partial D$. We want to show there exists a ball B, with

$$z_0 \in \overline{B}, \qquad B \subset D. \tag{4.4}$$

If $z_0 \in \partial D_k$, then (4.4) follows directly from the induction hypothesis. If $z_0 \notin \partial D_k$, then there exists a sequence of points $\{z_j\} \subset L_{k+1} \setminus \overline{D}_k$ which converges to z_0. For each j choose a ball $B_j(y_j, r_j)$ which lies in L_{k+1}, intersects L_k, and contains z_j. By passing to subsequences we may assume that $\{y_j\}$ and $\{r_j\}$ converge. Setting $r_0 = \lim r_j$ and $y_0 = \lim y_j$, we have

$$|z_0 - y_0| \le |z_0 - z_j| + |z_j - y_j| + |y_j - y_0|;$$

hence given any $\varepsilon > 0$ there exists an M such that

$$|z_0 - y_0| \le r_0 + \varepsilon$$

for all $j > M$. We conclude $z_0 \in \overline{B(y_0, r_0)}$. If $r_0 > 0$ we have (4.4); hence we assume that $r_0 = 0$ or $y_0 = z_0$. To show $r_0 = 0$ implies $z_0 \in \partial D_k$, we let $\varepsilon > 0$ be given and choose j such that both $|y_j - z_0| < \varepsilon/3$ and $r_j < \varepsilon/3$. Then $B(y_j, r_j) \subset B(z_0, \varepsilon)$, so that $B(z_0, \varepsilon)$ intersects L_k for all $\varepsilon > 0$, i.e. $z_0 \in \partial D_k$. □

Domains with outwardly directed corners or spires are not oscillation domains. In particular the slit domain formed from a disc by removing an arc containing a right angle is not an oscillation domain, even though it does satisfy the conclusion of Theorem 4.3. The following theorem provides a sufficient condition.

4.5. Theorem. *Let D be a domain in $\mathbf{R}^n$ and suppose each point in D lies in a ball $B \subset D$ of fixed radius $\delta > 0$. Then D is an oscillation domain.*

Proof. We show D has the chaining property of Theorem 4.1 with $N = N\big(n, m(D), \delta\big)$. Consider the cover Γ of D given by taking all balls $B \subset D$ such that the radius of $B \geq \delta$. Let $\tau < 1$ be a constant, define

$$E = \bigcup_{B\in\Gamma} \tau B, \tag{4.6}$$

and let m denote the number of components of E. As each component of E contains at least one ball of radius greater than or equal to $\tau\delta$,

$$m \leq \frac{m(D)}{\tau^n \delta^n \omega_n}. \tag{4.7}$$

We show now that the set E can be covered by M balls $B \subset D$ where $M = M\big(n, m(D), \delta\big)$. Observe first that

$$E \subset \big\{x \in D \bigm| d(x, \partial D) \geq r\big\}, \qquad \text{where} \quad r = (1-\tau)\delta. \tag{4.8}$$

For each point $x \in E$, we define the ball $B(x) = B(x, r)$. By (4.8) $B(x) \subset D$ and the union of all such balls covers E. We again use the covering lemma [S] as in the proof of Lemma 2.19 to extract a subcover $\{B_j\}$ such that

$$\sum 5^{-n} m(B_j) = \sum 5^{-n} r^n \omega_n \leq m(D). \tag{4.9}$$

Thus $\{B_j\}$ contains at most M balls, where

$$M \leq \left(\frac{5}{r}\right)^n \left(\frac{m(D)}{\omega_n}\right). \tag{4.10}$$

Denote the components of E by E_1, ..., E_m and define

$$F_i = \bigcup_{B\subset E_i} \frac{1}{\tau}(B). \tag{4.11}$$

The union of the sets F_i covers D by hypothesis and since D is connected each F_i must intersect some F_j for $i \neq j$. Observe that if F_i intersects F_j, then we can find balls B_i, B_j in D such that B_i intersects E_i, B_j intersects E_j, and B_i intersects B_j; this gives us a means to chain across the components of E.

Consider any two points x and y in D and take for B_0 and B_s any balls of radius δ containing x and y respectively. Then B_0 intersects E_i for some i and B_s intersects E_j for some j. Connecting component E_i to E_j requires at most $2m$ balls and E itself can be covered by M balls. Therefore a chain B_0, B_1, ..., B_s exists joining x to y with

$$s \leq 2m + M + 1. \ \square \tag{4.12}$$

The condition in Theorem 4.8 is not necessary as can be seen in the following example.

4.13. Example. Let B_0 be any ball, take a sequence of disjoint balls B_j intersecting B_0, but not contained in B_0, whose radii tend to zero and define $D = \cup B_j$. Clearly D has the covering property of Theorem 4.1 with $N = 2$, but the hypothesis of Theorem 4.5 is not satisfied. □

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University of Texas
Department of Mathematics
Austin, TX 78712
U.S.A.

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