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THE MIXED ELLIPTICALLY FIXED POINT PROPERTY FOR KLEINIAN GROUPS

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Abstract. We define the mixed elliptically fixed point (MEFP) property for Kleinian groups. Such a property is a set of possibilities for the locations on the Riemann sphere of the fixed points of elliptic elements in Kleinian groups. We show that the MEFP property is invariant under the operations given by the Maskit–Klein combination theorems. As a consequence, we obtain that finitely generated function groups satisfy such a property. We also show that geometrically finite Kleinian groups satisfy the MEFP property. Examples of Kleinian groups where such a property does not hold are provided.

1. Introduction

We consider a set of possible locations for the fixed points of elliptic elements in Kleinian groups. We call such a set of possibilities the mixed elliptically fixed point (MEFP) property for Kleinian groups. We show that the MEFP property is invariant under the operations of the Klein–Maskit combination theorems (Theorems C1 and C2). This result and Maskit's classification theorem of finitely generated function groups given in [3] imply that these groups satisfy the MEFP property (Theorem A).

The fact that geometrically finite Kleinian groups have finite-sided convex fundamental polyhedron in the hyperbolic three space implies the desired result for this class of groups (Theorem B).

In general, a Kleinian group G does not always satisfy the MEFP property. We provide a couple of groups for which this property fails. In fact, finite normal extensions of Kleinian groups satisfying the MEFP property do not necessarily satisfy such a property (see the remark at the end of Section 5). Corollaries 1 and 2 assert that this is not the case if we consider finite extensions of finitely generated function groups.

The MEFP property can be used to study the problem of lifting conformal automorphisms of Riemann surfaces to the region of discontinuity of Kleinian groups uniformizing it. In fact, this work started from an attempt to give necessary and sufficient conditions for a group H of conformal automorphisms of a closed Riemann surface S to be lifted to the region of discontinuity of some Schottky

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group G uniformizing S. Corollary 1 of this paper has been applied in that direction [2].

This work is organized as follows. In Section 2 we recall some definitions which can also be found in [4]. In Section 3 we define the MEFP property. We give examples where such a property holds and provide two examples where it does not. In Section 4 we recall the Klein–Maskit combination theorems and Maskit's classification theorem of finitely generated function groups [3]. In Section 5 we state the main theorems and corollaries of this work. They indicate that the MEFP property is invariant under the combination theorems and that a Kleinian group which is either a finitely generated function group or a geometrically finite Kleinian group satisfies the MEFP property. We apply these results to finite extensions of finitely generated function groups (Corollaries 1 and 2). In Section 6 we prove Theorems C1, C2 and A. In Section 7 we prove Theorem B.

2. Preliminaries and definitions

A Kleinian group G is a subgroup of the group $\mathbf{M}$ of Möbius transformations (also called fractional linear transformations), which acts discontinuously on some part of the Riemann sphere. The group $\mathbf{M}$ is canonically isomorphic to the projective group $PSL(2,\mathbf{C})$. The (open) set of points where G acts discontinuously is denoted by $\Omega(G)$ and called the region of discontinuity of G. The complement of $\Omega(G)$ is called the limit set of G and denoted by $\Lambda(G)$.

If T is a connected component of the region of discontinuity of the Kleinian group G, we say that T is a component of G.

Kleinian groups act in a natural way as orientation-preserving isometries of the hyperbolic three-space $\mathbf{H}^3 = \{\,(z,t); z \in \mathbf{C},\ t > 0\,\}$. The Riemann sphere $\hat{\mathbf{C}}$ can be thought of as the boundary of this space.

A Kleinian group G is said to be a function group if there exists a G-invariant component of its region of discontinuity. If G is also finitely generated, we say that G is a finitely generated function group.

A function group F for which the limit set is contained in some Jordan curve γ is called a quasi-Fuchsian group. A group G (necessarily a Kleinian group) which contains a quasi-Fuchsian group F of index two is called a $\mathbf{Z}/2\mathbf{Z}$-extension of the quasi-Fuchsian group F. In case we can choose the curve γ as an analytic circle, we call F a Fuchsian group.

Geometrically finite Kleinian groups are Kleinian groups with a finite-sided convex fundamental polyhedron in $\mathbf{H}^3$. We assume standard results about these groups; see for instance [4].

Degenerated groups are function groups with the property that the region of discontinuity is connected, simply-connected and hyperbolic, that is, its universal covering is the hyperbolic plane. We remark at this point that no explicit example is known of such a finitely generated group.

Kleinian groups which belong to the next families are called basic groups.

(1) finite groups;
(2) Euclidean groups;
(3) Kleinian groups with limit sets consisting of exactly two points;
(4) finitely generated quasi-Fuchsian groups;
(5) finitely generated degenerated groups.

3. The MEFP property on Kleinian groups

Definition (The MEFP property). A Kleinian group G is said to satisfy the mixed elliptically fixed point (MEFP) property if for any elliptic transformation $h \in G$ either:

(i) both fixed points of h belong to the region of discontinuity of G, $\Omega(G)$;
(ii) there exists a loxodromic transformation g in G commuting with h;
(iii) there exists a parabolic transformation p in G sharing one fixed point with h, and the other fixed point of h is in $\Omega(G)$;
(iv) there exist parabolic transformations p and q in G, each one sharing a fixed point with h; or
(v) G contains a degenerated subgroup H containing h.

Remarks. (1) In case (v), one of the fixed points of h, say x, necessarily belongs to the region of discontinuity of H and the other, say y, belongs to the limit set of H. In such a case, there exists neither a parabolic transformation in G fixing y nor a loxodromic transformation commuting with h [1]. (2) In a Kleinian group G satisfying the MEFP property, all the above possibilities (i)–(v) can occur for different elliptic elements of G. Examples of groups of these types can be easily constructed from the Klein–Maskit combination theorems (once Theorems C1 and C2 have been proved).

Examples of Kleinian groups having the MEFP property are:

(a) finite groups,
(b) Euclidean groups,
(c) Kleinian groups with limit sets consisting exactly of two points,
(d) quasi-Fuchsian groups,
(e) degenerated groups, and
(f) torsion-free Kleinian groups.

Groups of type (a), (b) and (c) are known as the elementary groups. They are easily seen necessarily to satisfy the MEFP property. The fixed points of elliptic elements in quasi-Fuchsian groups are always points in the region of discontinuity. In particular, they satisfy the above property. Cases (e) and (f) are trivial. As a consequence the basic groups satisfy the MEFP property. This observation will be used to prove Theorem A. The following are examples of Kleinian groups which do not satisfy MEFP property.

(g) This example is an easy modification of Maskit's example in [6] of a Web group which cannot be constructed from the combination theorems. Let H and H^* be finitely generated Fuchsian groups of the first kind, acting in the unit disc

$\triangle$. Assume H to be a purely hyperbolic group and $\triangle/H^*$ to be a Riemann surface of signature $(g,1;v)$, where $g \geq 1$ and $2 \leq v < \infty$. Assume that the origin, then ∞, is not an elliptic fixed point in H^*. Consider the transformation $a_t(z) = tz$, where t is a non-zero complex number, and let $H_t = a_t H a_t^{-1}$. Let G_t be the group generated by H_t and H^*. For sufficiently small $\|t\|$ the group G_t can be formed from H_t and H^* by Combination Theorem 1 (see Section 4); for each such t, G_t is the free product of H_t and H^* and does not contain parabolic elements. It can be shown that there exists a ray $\arg(t) = \theta_0$ on which every element of G_t is either loxodromic or elliptic. Assume $\theta_0 = 0$, that is, t is a real number. Let T be the set of all t such that $G_{t'}$ can be formed by the free combination in Maskit's sense, for all $t' \leq t$. It follows from Combination Theorem 1 that T is an open set. Let $t_0 = \operatorname{lub} T$ and let $G = G_{t_0}$. The group G is the free product of H_{t_0} and H^*, and every element of G is either loxodromic or elliptic. A fundamental set for G is the union of a fundamental set of H^* in the region $\{\|z\| > 1\}$ and a fundamental set of H_{t_0} in the disc $\{\|z\| < t_0\}$. Any component of G is conjugate to either $\{\|z\| < t_0\}$ or $\{\|z\| > 1\}$. G is actually a Web group (see [4] for a definition). Any elliptic element of H^* has one fixed point in the region $\{\|z\| > 1\}$, that is, it is a regular point of the group G. The other fixed point is a limit point of G. Since G has neither parabolic elements nor degenerated subgroups, G cannot satisfy the MEFP property.

(h) The following example is an infinitely generated Kleinian group. Let G be the group generated by the transformations A, B_1, B_2, ..., where

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B_n = \begin{pmatrix} 4n & (16n^2-1)/4 \\ 4 & 4n \end{pmatrix}.$$

The group G is a Kleinian group without parabolic elements as a consequence of the Maskit–Klein combination theorems [4] (also Section 4). The elliptic element A has one fixed point, the origin, in the region of discontinuity of G and the other fixed point, ∞, belongs to the limit set of G. The group G does not contain degenerated subgroups. In particular, G does not satisfy the MEFP property.

4. The Klein–Maskit combination theorems and function groups

Let H be a subgroup of the Kleinian group G. A set T is called precisely invariant under H if:

(i) $h(T) = T$ for every h in H; and

(ii) $g(T) \cap T = \phi$ for all g in $G - H$.

For a cyclic subgroup H of G, a precisely invariant topological disc B is the interior of a closed topological disc, with closure $\bar{B}$, where $\bar{B} - \Lambda(H)$ is precisely invariant under H, and $\big(\bar{B} - \Lambda(H)\big) \subset \Omega(G)$. Here $\Lambda(H)$ represents the limit set of the subgroup H.

Let γ_i be a sequence of loops and let z be a point on the Riemann sphere. We say that the sequence of loops γ_i nests about z if the loop γ_i separates z from the loop γ_{i-1} and any sequence of points $q_i \in \gamma_i$ converges to z. We permit the point z to belong to some γ_{i_0}, in which case z must belong to γ_i, for all $i \geq i_0$.

Combination Theorem 1. *Let G_1, G_2 be Kleinian groups with a common subgroup H. Assume the group H is either trivial, or cyclically generated by an elliptic or a parabolic transformation. For $i = 1, 2$, let B_i be a precisely invariant topological disc under H in G_i. Assume that B_1 and B_2 have a common boundary γ and $B_1 \cap B_2 = \phi$. Let G be the group generated by G_1 and G_2; then*

(1) *G is Kleinian.*
(2) *$G = G_1 *_H G_2$, that is, G is the free product of G_1 and G_2 with an amalgamated subgroup H.*
(3) *For $z \in \Lambda(G)$ where z is not a limit point of a conjugate of either G_1 or G_2, there exists a sequence $\{j_n\}$ of elements of G so that $\{j_n(\gamma)\}$ nests about z.*
(4) *If γ is precisely invariant under H in either G_1 or G_2, then every elliptic or parabolic element in G lies in a conjugate of either G_1 or G_2.*

Combination Theorem 2. *Let G_1 be a Kleinian group and let H_1 and H_2 be subgroups of G_1. Assume that the groups H_1 and H_2 are either trivial, or cyclically generated by an elliptic or parabolic transformation. For $i = 1, 2$, let B_i be a precisely invariant topological disc under H_i in G_1, and let γ_i be the boundary of B_i. We assume that $g(\bar{B}_1) \cap \bar{B}_2 = \phi$, for all g in G_1. Let $G_2 = \langle f \rangle$ be a cyclic group generated by f, where $f(\gamma_1) = \gamma_2$, $f(B_1) \cap B_2 = \phi$, and $f \circ H_1 \circ f^{-1} = H_2$. Let G be the group generated by G_1 and G_2; then*

(1) *G is Kleinian.*
(2) *$G = G_1 *_{G_2}$, that is, every relation in G is a consequence of the relations in G_1 and the relations $f \circ H_1 \circ f^{-1} = H_2$.*
(3) *For every point $z \in \Lambda(G)$ where z is not a limit point of a conjugate of either G_1 or G_2, there exists a sequence $\{j_n\}$ of elements of G so that $\{j_n(\gamma_1)\}$ nests about z.*
(4) *If γ_1 is precisely invariant under H_1 in G_1, then every elliptic or parabolic element in G lies in a conjugate of G_1.*

From now on, we assume the following additional conditions on the parabolic cyclic groups in the combination theorems.

(I) In Combination Theorem 1, assume that if H is a cyclic group generated by a parabolic transformation, H must also be its own normalizer in either G_1 or G_2. This means that conclusion (4) of Combination Theorem 1 can be applied.
(II) In Combination Theorem 2, assume that if H_1 and H_2 are cyclic groups generated by parabolic transformations, then each of them is its own normalizer in G_1. This means that conclusion (4) of Combination Theorem 2 can be applied.

A proof of the combination theorems can be found in Chapter VII of [4]. Another proof of these theorems (as established above) is given in [3].

Classification Theorem [3]. *Let G be a finitely generated function group. Then G can be constructed from the basic groups using Combination Theorems* 1 *and* 2 *under the assumptions* (I) *and* (II).

5. Main results

In this section we establish the main theorems of this work and two consequential corollaries.

Theorem A. *Finitely generated function groups satisfy the MEFP property.*

Theorem B. *Geometrically finite Kleinian groups satisfy the MEFP property.*

Remark. In Theorem A the condition "finitely generated" is necessary (cf. Example (h)). Since geometrically finite Kleinian groups do not contain degenerated groups [3], the case (v) in the definition of the MEFP property cannot occur in these cases.

Examples of geometrically finite function groups without parabolic elements are the finite extensions of Schottky groups (see [5] for a definition of Schottky groups).

Corollary 1. *Let G be a finite extension of a Schottky group and let h be any elliptic transformation in G. Then the only possibilities for the fixed points of h are given by* (i) *and* (ii) *in the definition of the MEFP property.*

Corollary 2. *Let $\tilde{G}$ be a finite extension of a finitely generated function group G. Then $\tilde{G}$ satisfies the MEFP property.*

Proof. Let h be an elliptic element in $\tilde{G}$, and let $\triangle$ be a G-invariant component of $\Omega(\tilde{G}) = \Omega(G)$. We have two possibilities for the image of $\triangle$ under the transformation h.

Case 1. $h(\triangle) = \triangle$. In this case the group G_1, generated by G and h, is also a function group. Since G_1 is a finite extension of a finitely generated group, it is also finitely generated. Theorem A now implies that G_1 satisfies the MEFP property. Since G_1 is a subgroup of $\tilde{G}$ of finite index, it is easy to see that h must satisfy some of the conditions in the definition of the MEFP property.

Case 2. $h(\triangle) \neq \triangle$. Let us denote by $\triangle_1$ the image of $\triangle$ under the transformation h. Then $\triangle_1$ is another component of the group $\tilde{G}$. Let F be the subgroup of $\tilde{G}$ keeping $\triangle$ invariant. Necessarily, F also keeps $\triangle_1$ invariant. Now F is a finitely generated Kleinian group with two invariant components. It is well known [5] that such a group is a quasi-Fuchsian group. In particular, the components are simply connected with a quasi-circle as a common boundary. The group $\tilde{G}$ is then a $Z/2Z$-extension of the finitely generated quasi-Fuchsian group F. As a consequence, $\tilde{G}$ is a geometrically finite Kleinian group. It follows from Theorem B that $\tilde{G}$ satisfies the MEFP property. □

Remark. Finite normal extensions of finitely generated Kleinian groups with the MEFP property do not generally satisfy such a property. As an example, consider a normal torsion-free subgroup F of finite index (Selberg's lemma) of the group G constructed in (h). The group F trivially satisfies the MEFP property, since it is torsion-free.

The next two theorems show the invariance of the MEFP property under the Klein–Maskit combination theorems. They will be used to prove Theorem A.

Theorem C1. *Under the same hypotheses as in Combination Theorem* 1*, assume that* G_1 *and* G_2 *satisfy the MEFP property. Then the group* G*, generated by* G_1 *and* G_2*, satisfies the MEFP property.*

Theorem C2. *Under the same hypotheses as in Combination Theorem* 2*, assume that* G_1 *satisfies the MEFP property. Then the group* G*, generated by* G_1 *and* $G_2 = \langle f \rangle$*, satisfies the MEFP property.*

6. Proof of Theorems C1, C2 and A

Proof of Theorem C1. Let t be an elliptic transformation of G. Conclusion (4) in Combination Theorem 1 implies that t is conjugate to an elliptic transformation h in G_1 or G_2. Without loss of generality, we may assume h to be in G_1. Let us observe that t satisfies any of the conditions of the definition of the MEFP property if and only if h does. In particular, we only need to check that h satisfies any of these conditions. Denote the fixed points of h by x and y. We have that either

(1) h is not a G-conjugate of any element of H, or
(2) h is a G-conjugate of some element in H.

Case (1) Let us assume h is not a G-conjugate of any element of H. Since $\bar{B}_1$ is a precisely invariant topological disc under H in G_1 and h is not a G-conjugate of any element of H, the fixed points of h necessarily belong to $B_2 - \cup\{g(\bar{B}_1);\ g \in G_1\}$. In particular, x and y are regular points of G_2. Assume x and y to be limit points of G_1. Then, by the MEFP property of G_1, either there exists a loxodromic transformation g in G_1 with x and y as fixed points, or there exist parabolic transformations P and Q in G_1 with fixed points x and y, respectively, or there exists a degenerated subgroup H_1 of G_1 containing h. Since G_1 is a subset of G, this case is proved. Let us assume x and y to be regular points of G_1. Conclusion (3) of Combination Theorem 1 implies that x and y are regular points of G. Now assume that one of the fixed points of h, say x, is a regular point of G_1 and the other is a limit point of G_1. Again, conclusion (3) of Combination Theorem 1 implies that x is a regular point of G. The MEFP property of G_1 implies that either there exists a parabolic transformation Q in G_1 with y as a fixed point, or there exists a degenerated subgroup H_1 of G_1 containing h (y is then a limit point of H_1). Since G_1 is contained in G, we are done.

Case (2) Assume h is G-conjugated to some element of H. In this case H must be a cyclic group generated by an elliptic transformation. Let j be a generator of the group H. Without loss of generality, we may assume h to be some power of j. It is easy to check that a non-trivial power of j satisfies some of the conditions given in the definition of the MEFP property if and only if j does, so we assume $h = j$, with x and y as its fixed points. Clearly, one of the fixed points of j, say x, belongs to B_1 and the other belongs to B_2. In this case x is a regular point of G_1 and y is a regular point of G_2. Since G_1 and G_2 satisfy the MEFP property, we only have the following possibilities:

(a1) y is a regular point of G_1, or
(a2) there exists a parabolic transformation $g_1 \in G_1$ with y as a fixed point, or
(a3) G_1 contains a degenerated subgroup H_1 containing j; and either
(b1) x is a regular point of G_2, or
(b2) there exists a parabolic transformation $g_2 \in G_2$ with x as fixed point, or
(b3) G_2 contains a degenerated subgroup H_2 containing j.

We have nine possibilities, and since G_1 and G_2 are subsets of G, the following lemma concludes the proof. □

Lemma 1. (1) (a1) *implies that y is either a regular point of G, or there exists a loxodromic transformation in G commuting with j.* (2) (b1) *implies that x is either a regular point of G, or there exists a loxodromic transformation in G commuting with j.*

Proof. The proofs of (1) and (2) are the same. We only prove part (1). Let us assume y is a limit point of G. Conclusion (3) of Combination Theorem 1 implies that there exists a sequence of G-translates of the loop γ nesting about y. Since y belongs to B_2, it follows that there exists g_1 in $G_1 - H$ such that y belongs to $g_1(B_1)$. In this case, $g_1^{-1}(y)$ belongs to B_1 and it is a fixed point of $g_1^{-1} \circ j \circ g_1 \in G_1$. Since B_1 is precisely invariant under H in G_1 and $g^{-1}(y)$ is different from y, we have $g^{-1}(y) = x$ and $g^{-1}(x) = y$, that is, $g_1(x) = y$ and $g_1(y) = x$. In particular, x is also a limit point of G and g_1 has order two. Applying the same argument to x, we can find an element g_2 in $G_2 - H$ such that $g_2(x) = y$, $g_2(y) = x$ and g_2 has order two. Consider the transformation $g = g_2 \circ g_1$. It can be seen that g is necessarily a loxodromic transformation in G with x and y as fixed points. In particular, g commutes with j. □

Proof of Theorem C2. Let t be an elliptic transformation of G. Conclusion (4) of Combination Theorem 2 implies that t is conjugated to an elliptic transformation h in G_1. As observed in the proof of Theorem C1, we may assume $t = h$. Denote by x and y the fixed points of the transformation h. We have the following possibilities for h; either

(1) h is not G-conjugated to any element in H_1, or

(2) h is G-conjugated to some element in H_1. (In this case H_1 is necessarily a cyclic group generated by an elliptic transformation.)

Case (1) Assume h is not a G-conjugate to any element of H_1. Since $\bar{B}_1$ is precisely invariant under H_1, $\bar{B}_2$ is precisely invariant under H_2 and h is not a G-conjugate to any element in H_1, the points x and y must belong to $\hat{\mathbf{C}} - \{\cup(g(\bar{B}_1) \cup g(\bar{B}_2)); g \in G_1\}$. Assume now x and y to be limit points of G_1. The MEFP property of G_1 asserts that either there exists a loxodromic transformation g in G_1 commuting with h, or there are parabolic transformations P and Q in G_1 with x and y as fixed points, respectively, or there exists a degenerated subgroup H_0 in G_1 containing h. Since G_1 is a subset of G, this case is proved. Assume now x and y to be regular points of G_1. Since x and y are not in $\bar{B}_1 \cup \bar{B}_2$, they cannot be fixed points of f. In particular, they are regular points of G_2. Conclusion (3) (of Combination Theorem 2) implies that x and y are regular points of G, and this case is clear. Assume now that one of the fixed points of h, say y, is a limit point of G_1 and x is a regular point of G_1. The MEFP property of G_1 implies that either there is a parabolic transformation g in G_1 with y as a fixed point, or there exists a degenerated subgroup H_0 in G_1 containing h. Since x does not belong to $\bar{B}_1 \cup \bar{B}_2$, the point x cannot be a fixed point of f. In particular, x is a regular point of G_2. Again, conclusion (3) of Combination Theorem 2 implies that x is a regular point of G. In this case either condition (iii) or (v), in the definition of the MEFP property, holds for the group G.

Case (2) Assume h is G-conjugated to some element of H_1. In this case H_1 is the cyclic group generated by the element j. As in case (2) in the proof of Theorem C1, we may assume $h = j$. As before, x and y denote the fixed points of j. One of these points must belong to B_1, say x. Since $f \circ H_1 \circ f^{-1} = H_2$, we have $f(y)$ belonging to B_2. Note that the fixed points of $f \circ j \circ f^{-1}$ are $f(x)$ and $f(y)$. In particular, x and $f(y)$ are regular points of G_1. The MEFP property of G_1 implies the following (only) possibilities:

(a1) y is a regular point of G_1, or

(a2) there exists a parabolic transformation $g_1 \in G_1$ with y as a fixed point, or

(a3) there exists a degenerated subgroup H_0 in G_1 containing j.

Case (a3) is clear, since G_1 is contained in G. From now on, assume that there is no degenerated subgroup of G containing j. In particular, there is no degenerated subgroup of G containing $f \circ j \circ f^{-1}$. The MEFP property of G_1 implies that:

(b1) $f(x)$ is a regular point of G_1, or

(b2) there is a parabolic transformation $g_2 \in G_1$ with $f(x)$ as a fixed point. In this case x is also a parabolic fixed point.

The next lemma concludes the proof, with the observation that the four possibilities (a1) and (b1); (a1) and (b2); (a2) and (b1); (a2) and (b2) each gives one of the possibilities in the definition of the MEFP property. □

Lemma 2. (1) (a1) *implies that either* y *is a regular point of* G *or there exists a loxodromic transformation* g *in* G *commuting with* j. (2) (b1) *implies that either* x *is a regular point of* G *or there exists a loxodromic transformation* g *in* G *commuting with* j.

Proof. Note that (2) is equivalent to the following: (2′) (b1) implies that either $f(x)$ is a regular point of G or there exists a loxodromic transformation g in G commuting with $f \circ j \circ f^{-1}$.

The proof of (2) is the same as the proof of (1), so we only prove (2). Assume x is a limit point of G. Since f belongs to G, the point $f(x)$ is also a limit point of G. Conclusion (3) of Combination Theorem 2 and the fact that $f(x)$ is a regular point of G_1 imply that there exists $g_1 \in G_1 - H_1 \cup H_2$ such that $f(x)$ belongs to $g_1(B_1)$. The fact that B_1 is precisely invariant under H_1 in G_1 and the fixed point $g_1^{-1}\big(f(x)\big)$ of the transformation $k = g_1^{-1} \circ f \circ j \circ f^{-1} \circ g_1$ belongs to B_1, implies that k is in H_1. Since the transformation k is not the identity and H_1 is a cyclic group generated by j, the fixed points of k must be x and y. The only possibility for this to happen is $g_1(x) = f(x)$ and $g_1(y) = f(y)$. We consider the transformation $g = f^{-1} \circ g_1$ which is loxodromic with x and y as fixed points, so commuting with j. □

Proof of Theorem A. The classification theorem asserts that any finitely generated function group is constructed from the basic groups using Klein–Maskit combination theorems. Now, Theorems C1 and C2 and the fact that the basic groups satisfy the MEFP property imply the desired result. □

7. The MEFP property and geometrically finite Kleinian groups

In this section we prove Theorem B. Let us start with some properties of geometrically finite Kleinian groups which will be used in the proof.

The equivalence of the following statements is shown in Chapter VI of [4].

(1) G is geometrically finite.
(2) Any convex fundamental polyhedron for G in $\mathbf{H}^3$ is a finite-sided one.
(3) Any limit point of G is either a point of approximation or a double-cusped parabolic fixed point or a rank-two parabolic fixed point.

Proof of Theorem B. If G is torsion free, there is nothing to check. Assume now G has torsion. Let $h \in G$ be any elliptic element with fixed points x and y.

Case (1) x and y are regular points of G, in which case we are done.

Case (2) x or y is a limit point. Without loss of generality we can assume that y is a limit point. Let $j \in G$ be be a primitive elliptic element with y as a fixed point.

We claim the following :

(i) Either $j(x) = x$ or there is a parabolic element in G with y as a fixed point.
(ii) If $g(y) = y$, for g in G, then either g is conjugate to a power of j in G, or g is a loxodromic element with x and y as fixed points, or there is a parabolic

element with y as its fixed point.

In fact, (i) if we assume $j(x) \neq x$, the commutator $[j,h] = j \circ h \circ j^{-1} \circ h^{-1}$ is a parabolic element in G with y as a fixed point. (ii) Let g be an element in G such that $g(y) = y$. Assume y is not a fixed point of any parabolic element of G. The only possibility for g is to be either an elliptic or a loxodromic element. By our assumption on y, necessarily $g(x) = x$, otherwise $[g,j]$ will be a parabolic element of G with y as a fixed point. At this point, g is either a power of j or a loxodromic element with x and y as fixed points. This ends the proof of our claim.

From now on we assume there is no parabolic element of G with y as a fixed point. Let L be the hyperbolic geodesic in $\mathbf{H}^3$ with x and y as its end points. Then j fixes L pointwise. Let P be a convex fundamental polyhedron for G. Since y is a limit point which is not a parabolic fixed point, y must be a point of approximation for G (see p. 128 in [4]). This implies that y cannot be in the closure of P (see p. 122 of [4]). As a consequence, we can find a sequence of points $y_n \in L$, converging to y, all of them non-equivalent points by G, and a sequence $g_n \in G$, $g_n \neq g_m \circ j^l$, for all l and $n \neq m$, such that $g_n(y_n) = z_n \in \bar{P}$, where $\bar{P}$ denotes the Euclidean closure of P. Let us consider a subsequence such that z_n converges, say to z, $g_n(y)$ converges, say to u, and $g_n(x)$ converges, say to t. In this way, the points u and t are limit points of the group G. Since $z_n \in \bar{P}$, we must also have $z \in \bar{P}$. We have two possibilities for z, that is, z is either a regular point of G, or z is a parabolic fixed point (see p. 128 in [4]). It is clear that z_n is an elliptic fixed point; in fact $z_n = g_n \circ j \circ g_n^{-1}(z_n)$. This implies that z_n must be on some edge of P. Since P has a finite number of edges, we can assume all the points z_n on the same edge of P. Let M be the geodesic in $\mathbf{H}^3$ containing this edge. In particular, z belongs to the closure of M.

We claim that z is a regular point of G. In fact, since z belongs to the boundary of the polyhedron P, either z is a regular point of G or z is a parabolic fixed point. If we assume z to be a parabolic fixed point, then the stabilizer of z in G (denoted by $\mathrm{stab}_G(z)$) is a Euclidean group [4]. All points in $\mathbf{H}^3$ are regular points of the group G, so z must be one of the end points of M. The discreteness of G implies the existence of a horoball H contained in $\mathbf{H}^3$ which is precisely invariant under $\mathrm{stab}_G(z)$ in G. In particular, z belongs to the boundary of H. The geodesic M must intersect H, and such an intersection is an arc of geodesic with one end point being z. Since the sequence of points z_n are in M and they converge to z, we can assume without loss of generality that all z_n belong to such an intersection. The linear transformations $g_n \circ j \circ g_n^{-1}$ have z_n as fixed points, so $g_n \circ j \circ g_n^{-1}(H) \cap H \neq \phi$. By the definition of H, we must have $g_n \circ j \circ g_n^{-1}(H) = H$ and $g_n \circ j \circ g_n^{-1}(z) = z$. The additional condition that $g_n \circ j \circ g_n^{-1}(z_n) = z_n$ implies $g_n \circ j \circ g_n^{-1}(w) = w$, for all w in the geodesic M. In particular, $g_n(L) = M$. Let us consider the linear transformations $h_n = g_n \circ g_{n+1}^{-1}$. These transformations have as fixed points the end points of M, in particular z, so they are not parabolic transformations. Since the transformation h_n cannot act as the identity on M by the choice of the sequence g_n, it must be loxodromic. This contradicts the fact that z is a parabolic fixed point and that G is a discrete group. In particular, z must be a regular point of G.

Let us consider the geodesics $L_n = g_n(L)$ through z_n having end points $g_n(x)$ and $g_n(y)$. Since $g_n(x)$ and $g_n(y)$ converge to t and u, respectively, L_n must converge either to a point or to the geodesic with end points u and t. If L_n converges to a point, we necessarily have $u = t = z$. This is a contradiction to the fact that z is a regular point and $u = t$ is a limit point for G. The other possibility is that L_n converges to a geodesic γ, with end points u and t. In this case, since the end points of γ are limit points of G and z is a regular point of G, we must have z in $\gamma \cap \mathbf{H}^3$. Any neighborhood of z contains z_n, for n sufficiently large. We know z is a regular point, so there exists a neighborhood U of it which is precisely invariant under $\mathrm{stab}_G(z)$, which is known to be finite. We can then assume without loss of generality that z_n belongs to U, $g_n \circ j \circ g_n^{-1}(z) = z$, and $g_n \circ j \circ g_n^{-1} = h$, for all n. In this case, we must have $(g_m^{-1} \circ g_n) \circ j \circ (g_m^{-1} \circ g_n)^{-1} = j$. Since $g_n \circ j \circ g_n^{-1}(z_n) = z_n$, $g_n \circ j \circ g_n^{-1}(z) = z$ and $z_n \neq z$, for all n, we must have $g_n \circ j \circ g_n^{-1}(w) = w$ for all w in γ. In particular, $g_n \circ j \circ g_n(t) = t$ and $g_n \circ j \circ g_n^{-1}(u) = u$. It follows that $\{g_n(x), g_n(y)\} = \{t, u\}$. The fact that $t \neq u$ and $g_n(x)$ converges to t implies that $g_n(x) = t$ and $g_n(y) = u$ for n sufficiently large. We may assume that the above holds for all n. The last statement implies that $g_m^{-1} \circ g_n(x) = x$ and $g_m^{-1} \circ g_n(y) = y$ for all n, m. The transformations $g_m^{-1} \circ g_n$ also keep L invariant, and for $n \neq m$ this transformation cannot be the identity on L. It follows that $g_m^{-1} \circ g_n$ is a loxodromic element in G with x and y as fixed points. □

References

[1] ANDERSON, J.: Mixing elements into Kleinian groups. - Ph.D. Thesis, S.U.N.Y. at Stony Brook, 1991.

[2] HIDALGO, R. A.: On Schottky groups with automorphism. - Ph.D. Thesis, S.U.N.Y. at Stony Brook, 1991.

[3] MASKIT, B.: Decomposition of certain Kleinian groups. - Acta Math. 130, 1973, 243–263.

[4] MASKIT, B.: Kleinian groups. - Grundlehren Math. Wiss. 287, Springer-Verlag, Berlin–Heildelberg–New York, 1988.

[5] MASKIT, B.: On extended quasifuchsian groups. - Ann. Acad. Sci. Fenn. Ser. A I Math. 15, 1990, 53–64.

[6] MASKIT, B.: Intersections of component subgroups of Kleinian groups. - In: Discontinuous groups and Riemann surfaces. Proceedings of the conference held in University Park, Md., 1973; L. Greenberg, editor. Ann. of Math. Stud. 79, Princeton Univ. Press, Princeton, N.J.

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