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A CONVERSE DEFECT RELATION FOR QUASIMEROMORPHIC MAPPINGS

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Abstract. Let $f\colon \mathbf{R}^n \to \bar{\mathbf{R}}^n$ be a nonconstant K-quasimeromorphic map. We prove first that given $C > 1$, there exists $\theta > 1$, θ depending only on n, K, C, such that whenever $a_1, \dots, a_q \in \bar{\mathbf{R}}^n$ are distinct, we have $n(r, a_j) \le CA(\theta r)$ for $j = 1, \dots, q$ and $r \in E$, where $E = E(f, a_1, \dots, a_q)$ has infinite logarithmic measure. This result is then used to obtain the following converse to the defect relation as established by S. Rickman. Let $f\colon \mathbf{R}^n \to \bar{\mathbf{R}}^n$ be a nonconstant K-quasimeromorphic map. Then there exist constants $C_1 > 1$ and $\theta_1 > 1$, depending only on n and K such that for $a_1, \dots, a_q \in \bar{\mathbf{R}}^n$ any distinct points, we have $\limsup_{\substack{r\to\infty \\ r \in E}} \sum_{j=1}^q ((n(r, a_j))/(A(\theta_1 r)) - 1)_+ \le C_1$ where E can be taken to be the same set as above. Any improvement or enlargement of the set E for the first result is immediately valid for the second (main) result.

1. Introduction

Quasiregular (and quasimeromorphic) mappings form a natural generalization of analytic (and meromorphic) maps to real n-dimensions. We abbreviate these classes as qr and qm. These functions retain some of the most important topological properties of analytic functions. A study of the value distribution theory of such maps has been a subject of interest for many years. For an overview of results in this area we refer to [R2].

Rickman has shown [R3] that a weak form of Picard's theorem holds for these mappings. Moreover in [R2], [R6] he proved that for a nonconstant, real n-dimensional, $n \ge 3$, K-qm function f, there exists a set $E \subset [1, \infty)$ of finite logarithmic measure, and a constant $C(n, K) < \infty$, depending only on n and K such that

$$\limsup_{\substack{r\to\infty \\ r \notin E}} \sum_{j=1}^q \Big(1 - \frac{n(r, a_j)}{A(r)}\Big)_+ \le C(n, K),$$

where $a_1, \dots, a_q$ are distinct points. For $n = 3$ this is qualitatively sharp, as can be seen from [R6, Theorem 1.7]. Thus Nevannlinna's defect relation generalizes in qualitative form to qm maps.

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In this paper, we consider a converse inequality. For a nonconstant meromorphic function f in the plane, it was shown by J. Miles [Mi] that there exist absolute constants $K<\infty$ and $C\in(0,1)$ and a set $E=E(f)\subset[1,\infty)$ having lower logarithmic density at least C such that if $a_1,\dots,a_q$ are distinct elements of the Riemann sphere, then

$$\limsup_{\substack{r\to\infty\\ r\in E}}\sum_{j=1}^{q}\Big(\frac{n(r,a_j)}{A(r)}-1\Big)_+\le K.$$

Here we extend the above result, for meromorphic functions in the plane, to qm maps and all dimensions.

The proof breaks up into two parts: Sections 3 and Section 4. In Section 3 we show that $n(r,a_j)\le CA(\theta r)$ for any given q points $a_1,\dots,a_q$ and r taking values in a set E of infinite logarithmic measure. This is an extension of [R1, 5.16], where the case $q=1$ is considered. The proof is a slight modification of the proof of the same. In Section 4 we first obtain an estimate which holds for all except possibly one value a_j. This estimate holds without the exceptional r-set, but the a_j chosen as exception does depend upon r. For such an a_j we then use the bound obtained in Theorem 3-1. An important open problem is to get a result such as Theorem 3-1 off an exceptional set which does not depend on a. The main analytic tool is path families, a natural generalization to space of extremal length.

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2. Notation and definitions

We denote by $\mathbf{R}^n$ the real euclidean n-space, and by $\bar{\mathbf{R}}^n$ the one-point compactification $\bar{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$. Set

$$B_r(x)=\{\,y\in\mathbf{R}^n:|x-y|<r\,\},\qquad S(x,r)=\partial B_r(x),$$
$$B(r)=B_r(0),\quad S(r)=S(0,r),\quad\text{and}\quad S=S(1).$$

The Lebesgue measure in $\mathbf{R}^n$ is denoted by $\mathscr{L}^n$ and the normalized k-dimensional Hausdorff measure in $\mathbf{R}^n$ by $\mathscr{H}^k$. We set $\omega_{n-1}=\mathscr{H}^{n-1}(S)$. The Euclidean metric in $\mathbf{R}^n$ is d. If $\gamma\colon\Delta\to\bar{\mathbf{R}}^n$ is a path, we denote its locus $\gamma\Delta$ by $|\gamma|$.

$\bar{\mathbf{R}}^n$ is equipped with the spherical metric,

$$d[x,y]=|x-y|/[(1+|x|^2)(1+|y|^2)]^{1/2};\qquad x,y\ne\infty$$
$$d[x,\infty]=1/(1+|x|^2)^{1/2}.$$

Definition. Let $n\ge 2$, and let G be a domain in $\mathbf{R}^n$. A continuous mapping $f\colon G\to\mathbf{R}^n$ is called *quasiregular* if (1) f is in the local Sobolev space $W^1_{n,\mathrm{loc}}(G)$;

i.e., f has distributional partial derivatives which are locally L^n-integrable, and (2) there exists a constant K, $1 \le K \le \infty$, such that

$$|f'(x)|^n \le K J_f(x) \tag{2-1}$$

holds for almost every $x \in G$. Here $|f'(x)|$ is the sup norm of the formal derivative $f'(x)$ defined by means of partial derivatives and $J_f(x)$ is the Jacobian determinant of f at x. The smallest K in (2-1) is the outer dilatation $K_O(f)$, and the smallest $K, 1 \le K \le \infty$, for which

$$J_f(x) \le K \inf_{|h|=1} |f'(x)h|^n \qquad \text{a.e.}$$

holds is the inner dilatation $K_I(f)$ of f. $K(f) = \max\bigl(K_O(f), K_I(f)\bigr)$ is the maximal dilatation of f. If f is quasiregular and $K(f) \le K$, it is called K-quasiregular.

Let $G \subset \bar{\mathbf{R}}^n$ be a domain. A mapping $f\colon G \to \bar{\mathbf{R}}^n$ is called *quasimeromorphic* if either $fG = \{\infty\}$ or the set $E = f^{-1}(\infty)$ is discrete and $f_1 = f|G \setminus (E \cup \{\infty\})$ is quasiregular. We set $K(f) = K(f_1), K_O(f) = K_O(f_1)$, and $K_I(f) = K_I(f_1)$.

For a definition of the modulus of a family of curves we refer to [Vu].

If $f\colon \mathbf{R}^n \to \bar{\mathbf{R}}^n$ is nonconstant and *qm,* the *counting function* $n(r,y)$ is defined for $r > 0$, $y \in \bar{\mathbf{R}}^n$, by

$$n(r,y) = \sum_{x \in f^{-1}(y) \cap \bar{B}(r)} i(x,f),$$

where $i(x,f)$ is the local topological index; see [MRV1].

$A(r)$ is the average of $n(r,y)$ over $\bar{\mathbf{R}}^n$ with respect to the spherical metric. If r, $t > 0$, $\nu\bigl(r, S(a,t)\bigr)$ is the average of the counting function over the sphere $S(a,t)$ with respect to $\mathscr{H}^{n-1}$,

$$\nu\bigl(r, S(a,t)\bigr) = \frac{1}{\omega_{n-1}} \int_S n(r, a+ty)\, d\mathscr{H}^{n-1}(y),$$
$$A(r) = \frac{2^n}{\omega_n} \int_{\mathbf{R}^n} \frac{n(r,y)}{(1+|y|^2)^n}\, dy.$$

In particular, when $S(a,t) = S(t)$, we set $\nu\bigl(r, S(t)\bigr) = \nu(r,t)$, and also $\nu(r,1) = \nu(r)$.

Let $f\colon G \to \bar{\mathbf{R}}^n$ be *qm.* A domain D such that $\bar{D} \subset G$ is called a normal domain if $f\partial D = \partial f D$. If $x \in G$ and U is a normal domain such that $U \cap f^{-1}\bigl(f(x)\bigr) = \{x\}$, then U is called a normal neighbourhood of x. By [MRV1, 2.10], every point in G has arbitrarily small normal neighbourhoods.

We repeatedly use the following result [R4, p. 228, 2.1]. If $\theta > 1$ and $r, s, t > 0$, then

$$\nu(\theta r, t) \geq \nu(r, s) - \frac{K_I |\log(t/s)|^{n-1}}{(\log\theta)^{n-1}}. \tag{2-2}$$

We also need a comparison between averages on non-concentric spheres, S and $S(a,t) \subset B(1/2)$, for t small enough, say $t < 1/4$. This can be obtained by applying the above result to the map $\phi \circ f$, where ϕ is a quasiconformal map of $\bar{\mathbf{R}}^n$ onto $\bar{\mathbf{R}}^n$, which is the translation $x \to x - a$ in $B_t(a)$ and it is the identity map outside $B(1)$. ϕ can be taken to be 4-bilipschitz. Thus we get,

$$\nu\big(r, S(a,t)\big) \leq \nu(2r) + c_1\big(\log(1/t)\big)^{n-1}, \tag{2-3}$$

where we may take $c_1 = 4^{2n-2}K/(\log 2)^{n-1}$, since ϕ is 4^{2n-2}-quasiconformal.

3. An upper bound on $n(r,a)/A(\theta r)$

Theorem 3-1. *Let $f\colon \mathbf{R}^n \to \bar{\mathbf{R}}^n$ be a nonconstant K-quasimeromorphic map. Then for each $C > 1$, there exists $\theta > 1$, $\theta = \theta(C, n, K)$, such that for every $a_1, \ldots, a_q \in \bar{\mathbf{R}}^n$, there exists a set $E = E(a_1, \ldots, a_q) \subset [1, \infty)$, with $\int_E d\lambda/\lambda = \infty$, such that*

$$n(r, a_j) \leq CA(\theta r) \qquad \text{for } j = 1, \ldots q, \quad r \in E. \tag{3-2}$$

Note that here the role of E is different from that in [R4]. We begin with an adaptation of [R1, 5.4] to the case that $a \neq 0$. It is a quantification of the fact that a nonconstant *qm* map is light.

Lemma 3-3. *Let $f\colon \mathbf{R}^n \to \bar{\mathbf{R}}^n$ be a nonconstant K-quasimeromorphic map. Choose $1 < u < v$, $t > 0$ and $r > 0$. Let $a \in \mathbf{R}^n$ be given. Set*

$$H_{a,f}(r,t) = \{\, \lambda \in [r, ur] : S(\lambda) \cap f^{-1}(B_t(a)^c) \neq \emptyset \,\}, \tag{3-4}$$

$$\phi_{a,f}(r,t) = \int_{H_{a,f}(r,t)} \frac{d\lambda}{\lambda}.$$

Then,

$$\nu\big(vr, S(a,t)\big) \geq \left[1 - \frac{2\omega_{n-1}K_I K_O}{c_n \phi_{a,f}(r,t)(\log v/u)^{n-1}}\right] n(r,a) \tag{3-5}$$

where $c_n > 0$ is the constant in [V1, 10.11] *which depends only on n.*

Proof. Using (2-2), we may obtain [R1, 5.5] without the constant c', as

$$\nu(vr,t) \geq \left[1 - \frac{2K_I K_O \omega_{n-1}}{c_n \phi(r,t)(\log v/u)^{n-1}}\right] n(r,0). \tag{3-6}$$

Let $g(z) = f(z) - a$. Then $\nu_g(vr,t) \equiv \nu_f\big(vr, S(a,t)\big)$ and $n_g(r,0) = n_f(r,a)$. Let $\zeta = w - a$, so that $g(z) = \zeta \circ f(z)$, and also $S(\lambda) \cap g^{-1}\big(B(t)^c\big) = S(\lambda) \cap f^{-1}(B_t(a)^c)$. Hence $H_{0,g}(r,t) = H_{a,f}(r,t)$ and $\phi_{0,g}(r,t) = \phi_{a,f}(r,t)$. Now (3-6) applied to g gives (3-5).

Proof of Theorem 3-1. We divide the proof into three steps. The second step proves the theorem under the normalization $a_1, \ldots, a_q \in B(1/2)$. The first and third steps are merely to facilitate this normalization.

Step I: Let $C > 1$ be given. Let $a_1, \ldots a_q \in \bar{\mathbf{R}}^n$. By a rotation of the sphere we may assume that $a_1, \ldots, a_q \in B(\tau/2)$ for some $\tau \geq 1$. Let $\sigma > 0$ be such that the balls $\{\bar{B}_{\sigma\tau}(a_j)\}$ are disjoint and $\{\bar{B}_{\sigma\tau}(a_j)\} \subset B(\tau/2)$ for all j. We claim that for given $r_0 > 0$, there exists $r_1 \geq r_0$ such that for all $r \in [r_1, u^{1/4} r_1]$,

$$n(r, a_j) \leq C A(\theta r) \qquad \text{for } j = 1, \ldots, q, \tag{3-7}$$

where $u > 1$ is defined in (3-11). By repeating this argument, we obtain our set $E = \cup_{i=1}^{\infty} [r_i, u^{1/4} r_i]$, so that E has infinite logarithmic measure. We may assume that $n(r_0, a_j) \geq 1$ for all j, since the j's for which $n(r, a_j) = 0$ for all r satisfy the claim. Let

$$C' = C^{1/4} > 1. \tag{3-8}$$

By [R1, 4.10] we choose r_0 so that for $r \geq r_0$,

$$\nu(r) < C' A(2r). \tag{3-9}$$

We assume ∞ is an essential singularity (i.e. f has no limit in $\bar{\mathbf{R}}^n$ as we approach ∞), for otherwise f extends to $\bar{\mathbf{R}}^n$ as a *qm* map and it has finite degree [MRV2], [MS]. By [R1, 3.1] we then have that $A(r) \to \infty$. So we may choose r_0 such that for $r \geq r_0$

$$C'^2 K_I \Big(\frac{\log \tau}{\log 2}\Big)^{n-1} + C' c_1 \Big(\log \frac{1}{\sigma}\Big)^{n-1} < (C'^4 - C'^3) A(r). \tag{3-10}$$

Step II: In this step we replace f by f/τ and $a_1, \ldots, a_q$ by $a_1/\tau, \ldots, a_q/\tau$. However, for convenience of notation, we still call them f and $a_1, \ldots, a_q$. Note that we are now in the situation $a_1, \ldots, a_q \in B(1/2)$, $\{\bar{B}_\sigma(a_j)\}$ disjoint and each $\bar{B}_\sigma(a_j) \subset B(1/2)$. In order to apply Lemma 3-3 we define $u > 1$ by

$$\frac{1}{C'} = 1 - \frac{4\omega_{n-1} K_O K_I}{c_n (\log u)^n} \tag{3-11}$$

where $c_n > 0$ is as in [V1, 10.11].

For $u > 1$, as above and $t, r > 0$, let $\phi_j(r,t) \equiv \phi_{a_j,f}(r,t)$ be as in Lemma 3-3, and let

$$\Psi(t) = \sup_{r \geq r_0} \min_{1 \leq j \leq q} \phi_j(r,t). \tag{3-12}$$

Then Ψ is decreasing in t.

Case (i): $\Psi(\sigma) \geq (7/8)\log u$.

Then, by the definition of $\Psi(\sigma)$, there exists $r_1 \geq r_0$ such that $\min_j \phi_j(r_1,\sigma) \geq (3/4)\log u$; i.e.

$$\phi_j(r_1,\sigma) \geq (3/4)\log u, \qquad 1 \leq j \leq q. \tag{3-13}$$

From the definition of $\phi_j(r_1,\sigma)$, we note that

$$\begin{aligned}\phi_j(r_1,\sigma) = \int_{H_j(r_1,\sigma)} \frac{d\lambda}{\lambda} &= \int_{H_j(r_1,\sigma)\cap[r_1,u^{1/4}r_1]} \frac{d\lambda}{\lambda} + \int_{H_j(r_1,\sigma)\cap[u^{1/4}r_1,ur_1]} \frac{d\lambda}{\lambda} \\ &\leq \frac{1}{4}\log u + \int_{H_j(r_1,\sigma)\cap[u^{1/4}r_1,ur_1]} \frac{d\lambda}{\lambda}.\end{aligned}$$

From this and (3-13) we obtain for $r \in [r_1, u^{1/4}r_1]$ and for all $j = 1, \ldots, q$,

$$\phi_j(r,\sigma) \geq \int_{H_j(r_1,\sigma)\cap[u^{1/4}r_1,ur_1]} \frac{d\lambda}{\lambda} \geq \frac{1}{2}\log u. \tag{3-14}$$

We now apply Lemma 3-3 with $a = a_j$, $t = \sigma$, $r \in [r_1, u^{1/4}r_1]$, $v = u^2$ along with (3-14) and (3-11) to obtain

$$\begin{aligned}\nu\big(vr, S(a_j,\sigma)\big) &\geq \Big[1 - \frac{2\omega_{n-1}K_I K_O}{c_n \phi_j(r,\sigma)(\log u)^{n-1}}\Big] n(r,a_j) \\ &\geq \Big[1 - \frac{4\omega_{n-1}K_I K_O}{c_n (\log u)^n}\Big] n(r,a_j) \\ &= \frac{1}{C'} n(r,a_j) \qquad j = 1, \ldots, q.\end{aligned} \tag{3-15}$$

Now using (2-3) with $t = \sigma$ and (3-15), we get for $r \in [r_1, u^{1/4}r_1]$ and $j = 1, \ldots, q$, that

$$n(r,a_j) \leq C'\nu\big(vr, S(a_j,\sigma)\big) \leq C'\nu(2vr) + C'c_1(\log 1/\sigma)^{n-1}. \tag{3-16}$$

Case (ii): $\Psi(\sigma) < (7/8)\log u$.

Since f is discrete, for each fixed r, $\phi_j(r,t) \to \log u$ as $t \to 0$. Let $t_0 = \inf\{\, t : t \le \sigma, \Psi(t) \le (7/8)\log u \,\}$. One checks that $t_0 > 0$. We may assume $t_0 < \sigma$. Let δ be so small that

$$0 < \delta < \min\{\tfrac{1}{2}t_0, \sigma - t_0\}, \qquad \frac{4\delta}{t_0} < (\log 2)\Big(\frac{C'-1}{K_I C'^2}\Big)^{1/(n-1)} \tag{3-17}$$

and let

$$t_1 = t_0 - \delta, \qquad t_1' = t_0 + \delta. \tag{3-18}$$

Since $\Psi(t_1) > \frac{7}{8}\log u$, there exists $r_1 \ge r_0$ with $\min_j \phi_j(r_1,t_1) \ge \frac{3}{4}\log u$; i.e.

$$\phi_j(r_1,t_1) \ge \tfrac{3}{4}\log u, \qquad j = 1,\dots,q.$$

From this we may conclude, exactly as in Case (i), that for $r \in [r_1, u^{1/4}r_1]$,

$$\phi_j(r,t_1) \ge \tfrac{1}{2}\log u, \qquad j = 1,\dots,q. \tag{3-19}$$

Now we apply Lemma 3-3 with $r \in [r_1, u^{1/4}r_1]$, $t = t_1$, $a = a_j$, $v = u^2$, along with (3-19) and (3-11), to obtain

$$\begin{aligned} \nu\big(vr, S(a_j,t_1)\big) &\ge \Big[1 - \frac{2\omega_{n-1}K_I K_O}{c_n \phi_j(r,t_1)(\log u)^{n-1}}\Big] n(r,a_j) \\ &\ge \Big[1 - \frac{4\omega_{n-1}K_I K_O}{c_n (\log u)^{n}}\Big] n(r,a_j) \\ &\ge \frac{1}{C'} n(r,a_j), \qquad 1 \le j \le q. \end{aligned} \tag{3-20}$$

Let $t_0 < t < t_1'$. By (3-12), $\Psi(t) \equiv \sup_{r \ge r_0} \min_j \phi_j(r,t) \le (7/8)\log u$, and since $2vr \ge r \ge r_0$, we find for an appropriate $1 \le l \le q$, that $\phi_l(2vr,t) \equiv \min_j \phi_j(2vr,t) \le (7/8)\log u$. Then by the definition of $\phi_l(2vr,t)$ there exists $\varrho \in [2vr, 2vur]$ such that $S(\varrho) \cap f^{-1}(B_t(a_l)^c) = \emptyset$. The analysis of [MRV1, 2.5], which is stated only for *qr* maps but applies as well to *qm* maps, shows that every component of $f^{-1}\big(B_t(a_l)^c\big)$ which meets $\bar B(\varrho)$ is a normal domain contained in $B(\varrho)$. Hence

$$n(\varrho, y) = n(\varrho, z) \qquad \text{for all } y, z \in \bar B_t(a_l)^c. \tag{3-21}$$

In particular, since $t < t_1' < \sigma$ and the $\{\bar B(a_j,\sigma)\}$ are disjoint, we have for $j \ne l$, $n(\varrho, y) = n(\varrho, a_j + t_1 y)$ for all $y \in S$. And so on averaging,

$$\nu(\varrho) = \nu\big(\varrho, S(a_j,t_1)\big) \qquad j \ne l. \tag{3-22}$$

For $j = l$, since $t < t_1'$, we note from (3-21) that $n(\varrho, y) = n(\varrho, a_l + t_1' y)$ for all $y \in S$. So again on averaging,

$$\nu(\varrho) = \nu\big(\varrho, S(a_l, t_1')\big). \tag{3-23}$$

We now replace $\nu\big(\varrho, S(a_l, t_1')\big)$ by $\nu\big(\varrho, S(a_l, t_1)\big)$ with controllable error. Letting $\theta = 2$, $s = t_1$, $t = t_1'$, $r = vr$, we obtain from (2-2) that

$$\nu\big(vr, S(a_l, t_1)\big) \le \nu\big(2vr, S(a_l, t_1')\big) + \frac{K_I\big(\log(t_1'/t_1)\big)^{n-1}}{(\log 2)^{n-1}}. \tag{3-24}$$

Now we find, using (3-18) and (3-17), that

$$\log\frac{t_1'}{t_1} = \log\Big(1 + \frac{2\delta}{t_0 - \delta}\Big) < \frac{2\delta}{t_0 - \delta} < \frac{4\delta}{t_0} < (\log 2)\Big(\frac{C' - 1}{K_I {C'}^2}\Big)^{1/(n-1)}.$$

Hence, from (3-24),

$$\nu\big(vr, S(a_l, t_1)\big) \le \nu\big(2vr, S(a_l, t_1')\big) + (C' - 1)/{C'}^2. \tag{3-25}$$

Since $n(r, a_l) \ge n(r_0, a_l) \ge 1$ as stated in Step I, we have from (3-20) that $\nu\big(vr, S(a_l, t_1)\big) \ge 1/C'$. Substituting this inequality on the right hand side of (3-25) and unraveling, we obtain,

$$\nu\big(vr, S(a_l, t_1)\big) \le C'\nu\big(2vr, S(a_l, t_1')\big).$$

But since $2vr \le \varrho \le 2vur$, the last inequality, together with (3-20) and (3-23) gives for $r \in [r_1, u^{1/4} r_1]$,

$$n(r, a_l) \le {C'}^2 \nu\big(\varrho, S(a_l, t_1')\big) = {C'}^2 \nu(\varrho). \tag{3-26}$$

And again using the fact that $2vr \le \varrho$ along with (3-20) and (3-22), we find for $j \ne l$, $r \in [r_1, u^{1/4} r_1]$

$$n(r, a_j) \le C'\nu\big(\varrho, S(a_j, t_1)\big) = C'\nu(\varrho). \tag{3-27}$$

Using the inequality $2vr \le \varrho$, we conclude in both cases, from (3-26), (3-27) and (3-16) that, for $j = 1, \dots, q$, $r \in [r_1, u^{1/4} r_1]$,

$$n(r, a_j) \le {C'}^2 \nu(\varrho) + C' c_1 (\log 1/\sigma)^{n-1}. \tag{3-28}$$

Finally, we recall the change of scale we made in the beginning of Step II, and conclude from (3-28) that for $r \in [r_1, u^{1/4} r_1]$,

$$n(r, a_j) \le {C'}^2 \nu(\varrho, \tau) + C' c_1 (\log 1/\sigma)^{n-1} \tag{3-29}$$

for the original f and $a_1, \dots, a_q$.

Step III: First we use (2-2) to replace $\nu(\varrho, \tau)$ by $\nu(2\varrho)$ in (3-29) and get

$$n(r, a_j) \le {C'}^2 \nu(2\varrho) + {C'}^2 K_I \Big(\frac{\log \tau}{\log 2}\Big)^{n-1} + C' c_1 (\log 1/\sigma)^{n-1}.$$

Using (3-9), (3-10) (3-8) and $\varrho \le 2uvr$ we now get for $r \in [r_1, u^{1/4} r_1]$ and $j = 1, \dots, q$,

$$n(r, a_j) \le {C'}^3 A(4\varrho) + ({C'}^4 - {C'}^3) A(4\varrho) \le CA(\theta r),$$

where $\theta = 8uv = 8u^3$. This proves the theorem.

4. The main result

We first prove an intermediate result, i.e., the estimate (4-2). This is an essential fact needed for the main theorem.

Theorem 4-1. *Let $n \geq 2$ and $K \geq 1$. There exist positive constants $\theta_0 = \theta_0(n,K)$, $b = b(n,K)$ such that if $f\colon \mathbf{R}^n \to \bar{\mathbf{R}}^n$ is a nonconstant K-qm map and $a_1,\dots,a_q \in \bar{\mathbf{R}}^n$, are any distinct points, with $q > 1$, then there exist $r_0 = r_0(a_1,\dots,a_q,f) > 0$ such that for each $r \geq r_0$, we have*

$$\sum_{\substack{j=1\\ j\neq J(r)}}^{q} n(r,a_j) \leq \Big[q + \frac{4K_I b}{(\log 2)^{n-1}} + 2\Big] A(16\theta_0 r), \tag{4-2}$$

for some $J(r) \in \{1,\dots,q\}$. The constants θ_0 and b are given by

$$\log\theta_0 = \frac{\omega_{n-1}K_O c_1}{2^{n-4}c_n n}, \qquad b = \frac{2K_O\omega_{n-1}}{c_n \log\theta_0} \tag{4-3}$$

with c_1 and c_n as in (2-3) *and* (3-5) *respectively.*

Observe that there is no exceptional set for the r-values here. However, the estimate obtained is close to what we want, save for one $a_{J(r)}$. For this $a_{J(r)}$ we use Theorem 3-1. We thus obtain our main result, Theorem 4-26, on the same exceptional set of r-values as that obtained in Theorem 3-1. It is worth noting that any enlargement or improvement of the set E of Theorem 3-1, is also valid for Theorem 4-26.

Proof of Theorem 4-1. Again we divide the proof into three steps with main body of the proof being in the second step.

Step I: We may assume, as in the proof of Theorem 3-1, that ∞ is an essential singularity, so that $A(r) \to \infty$ as $r \to \infty$. By a rotation we assume that $a_1,\dots,a_q \in \mathbf{R}^n$. Let $\tau \geq 1$ and $\sigma > 0$ be such that $B_{\sigma\tau}(a_j) \subset B(\tau/2)$, and the $\{\bar{B}_{\sigma\tau}(a_j)\}$ are disjoint. We set $r_0 = \max(r_1,r_2)$, where r_1 and r_2 are obtained below. Choose $r_1 = r_1(\tau,q,f) > 0$ such that for $r \geq r_1$,

$$\text{(4-4)}\qquad \begin{aligned} &\text{(i)} \quad \Big[q + \frac{K_I b}{(\log 2)^{n-1}}\Big] K_I \Big(\frac{\log\tau}{\log 2}\Big)^{n-1} \leq \frac{K_I b}{(\log 2)^{n-1}}\nu(r)\\ &\text{(ii)} \quad \nu(r) < \frac{q}{q-1}A(2r) \qquad \text{by [R1, 4.10].} \end{aligned}$$

Step II: Again by replacing f by f/τ we reduce to the case $\tau = 1$. Since

$\nu(r) \to \infty$ as $r \to \infty$, we can choose $r_2 = r_2(\sigma, q, f) > 0$ such that for $r \ge r_2$,

$$
\text{(4-5)}\qquad
\begin{aligned}
&\text{(i)} \quad [b\nu(2\theta_0 r)]^{1/n} + 1 < [2b\nu(2\theta_0 r)]^{1/n},\\
&\text{(ii)} \quad \log 2 < \bigl(b\nu(2\theta_0 r)\bigr)^{1/(n-1)} - (b\nu(2\theta_0 r))^{1/n},\\
&\text{(iii)} \quad \frac{1}{1 + \bigl(\log(\sigma/2)\bigr)/\bigl(b\nu(2\theta_0 r)\bigr)^{1/(n-1)}} < 2^{1/n},\\
&\text{(iv)} \quad 2\exp\bigl(-\tfrac{1}{2}\bigl(b\nu(2\theta_0 r)\bigr)^{1/n}\bigr) < \sigma,\\
&\text{(v)} \quad c_1 q b < \bigl(b\nu\bigl(2\theta_0 r\bigr)\bigr)^{1/n}.
\end{aligned}
$$

Fix $r \ge r_2$. Since f is qm, $\mathscr{H}^n(\partial B(\theta_0 r)) = 0$ implies $\mathscr{H}^n\bigl(f\bigl(\partial B(\theta_0 r)\bigr)\bigr) = 0$, by [Vu, 10.5(3)]. From this and Fubini's theorem it follows that $\mathscr{H}^{n-1}\bigl(f\bigl(\partial B(\theta_0 r)\bigr) \cap S(a_j, \sigma_1)\bigr) = 0$ for a.e.

$$\sigma_1 \in \bigl[\exp\bigl\{-\bigl(b\nu(2\theta_0 r)\bigr)^{1/(n-1)}\bigr\}, 2\exp\bigl\{-\bigl(b\nu(2\theta_0 r)\bigr)^{1/(n-1)}\bigr\}\bigr],$$

for each $j = 1, \ldots, q$. Hence there exists $\varepsilon_1 \in [1, 2]$ such that for

$$\text{(4-6)}\qquad \sigma_1 = \varepsilon_1 \exp\bigl\{-\bigl(b\nu(2\theta_0 r)\bigr)^{1/(n-1)}\bigr\}$$

$$\text{(4-7)}\qquad \mathscr{H}^{n-1}\bigl(f\bigl(\partial B(\theta_0 r)\bigr) \cap S(a_j, \sigma_1)\bigr) = 0 \qquad \text{for all } j = 1, \ldots, q.$$

Then by (4-6) and (4-5) (ii) we have

$$\text{(4-8)}\qquad \sigma_1 \le 2\exp\bigl\{-\bigl(b\nu(2\theta_0 r)\bigr)^{1/(n-1)}\bigr\} < \exp\bigl\{-\bigl(b\nu(2\theta_0 r)\bigr)^{1/n}\bigr\} = \sigma_2$$

and by (4-5) (iv),

$$\sigma_2 = \exp\bigl\{-\bigl(b\nu(2\theta_0 r)\bigr)^{1/n}\bigr\} < \sigma.$$

Let α_j and β_j be the maps of S onto $S(a_j, \sigma_1)$ and $S(a_j, \sigma_2)$ respectively given by $\alpha_j(y) = a_j + \sigma_1 y$, $\beta_j(y) = a_j + \sigma_2 y$.

For $y \in S$, let $\gamma_y^j\colon [0,1] \to \mathbf{R}^n$ be the line segment joining a_j to $\beta_j(y)$, parametrized so that $\gamma_y^j\colon [0,1/2]$ joins a_j to $\alpha_j(y) \in S(a_j, \sigma_1)$, $gyj\colon [1/2, 1]$ joins $\alpha_j(y)$ to $\beta_j(y) \in S(a_j, \sigma_2)$.

Comparison of $n(r, a_j)$ *with* $n\bigl(\theta_0 r, \alpha_j(y)\bigr)$: Let $f\,|\,X$ denote f restricted to X and let $\Lambda_y^j = \{\lambda_1, \ldots, \lambda_h\}$ be a maximal sequence of $f\,|\,B(4\theta_0 r + 1)$-liftings of $\gamma_y^j\,|\,[0, 1/2]$ starting at points of $f^{-1}(a_j) \cap \bar{B}(r)$, as defined in [R1]. Then necessarily $h = n(r, a_j)$. The following crucial lemma has been inspired by the proof of [R2, 3.2].

Lemma 4-9. *The family of curves*

$$\mathscr{F}_j = \bigcup_{y\in S}\Lambda_y^j$$

lies completely in $B(\theta_0 r)$*, except perhaps for one* $j = J(r) \in \{1,\dots,q\}$*.*

Proof. Note that by definition, all paths in $\mathscr{F}_j$ start at preimages of a_j in $\bar B(r)$. We prove the lemma by contradiction. Suppose there exist $j \neq k$ and $\eta_j \in \mathscr{F}_j$, $\eta_k \in \mathscr{F}_k$, such that η_j, $\eta_k \not\subseteq B(\theta_0 r)$. Let Γ be the family of paths in $B(\theta_0 r)\setminus \bar B(r)$ joining the loci $|\eta_j|$ and $|\eta_k|$. Note that $|f(\eta_j)|$ and $|f(\eta_k)|$ are line segments starting at a_j and a_k and contained in $\bar B(a_j,\sigma_1)$ and $\bar B(a_k,\sigma_1)$ respectively. Hence each path in $f\Gamma$ contains sub-paths which join $S(a_j,\sigma_1)$ to $S(a_j,\sigma)$ and $S(a_k,\sigma)$ to $S(a_k,\sigma_1)$. Set

$$\varrho(z) = \begin{cases} \big(2\log(\sigma/\sigma_1)|z-a_j|\big)^{-1}, & \sigma_1 < |z-a_j| < \sigma \\ \big(2\log(\sigma/\sigma_1)|z-a_k|\big)^{-1}, & \sigma_1 < |z-a_k| < \sigma \\ 0, & \text{otherwise.} \end{cases}$$

Then ϱ is well-defined by the choice of σ. Also, ϱ is admissible for the family $f\Gamma$, and by [MRV1, 3.2] we obtain

$$\begin{aligned} M(\Gamma) &\le K_O \int_{\mathbf{R}^n} \varrho(z)^n n(\theta_0 r, z)\, d\mathscr{L}^n(z) \\ &= \frac{K_O}{\big(2\log(\sigma/\sigma_1)\big)^n} \int_{\{\sigma_1<|z-a_j|<\sigma\}} n(\theta_0 r, z)|z-a_j|^{-n}\, d\mathscr{L}^n(z) \\ &\quad + \frac{K_O}{\big(2\log(\sigma/\sigma_1)\big)^n} \int_{\{\sigma_1<|z-a_k|<\sigma\}} n(\theta_0 r, z)|z-a_k|^{-n}\, d\mathscr{L}^n(z) \\ &= I + II. \qquad (4-10) \end{aligned}$$

We obtain an estimate for I. Exactly the same estimate holds for II as well. By transfering the integral of (4-10) into polar coordinates, we find that,

$$\begin{aligned} I &= K_O\big(2\log(\sigma/\sigma_1)\big)^{-n} \int_{\sigma_1}^{\sigma}\int_S n(\theta_0 r, a_j+\tau y)\, d\mathscr{H}^{n-1}(y)\tau^{-1}\, d\tau \\ &\equiv K_O\omega_{n-1}\big(2\log(\sigma/\sigma_1)\big)^{-n} \int_{\sigma_1}^{\sigma} \nu(\theta_0 r, S(a_j,\tau))\tau^{-1}\, d\tau. \end{aligned}$$

Using (2-3), with $\theta = \theta_0$,

$$\begin{aligned} I &\le \frac{K_O\omega_{n-1}}{\big(2\log(\sigma/\sigma_1)\big)^n} \int_{\sigma_1}^{\sigma} \big\{\nu(2\theta_0 r) + c_1\big(\log(1/\tau)\big)^{n-1}\big\}\tau^{-1}\, d\tau \\ &\le \frac{K_O\omega_{n-1}}{\big(2\log(\sigma/\sigma_1)\big)^n}\Big[\nu(2\theta_0 r)\log(\sigma/\sigma_1) + c_1\frac{\big(\log(1/\sigma_1)\big)^n}{n}\Big] \qquad (4\text{-}11) \\ &\le \frac{K_O\omega_{n-1}}{2^n}\Big[\frac{\nu(2\theta_0 r)}{\big(\log(\sigma/\sigma_1)\big)^{n-1}} + \frac{c_1}{n}\Big(\frac{\log 1/\sigma_1}{\log\sigma/\sigma_1}\Big)^n\Big] \end{aligned}$$

Now using (4-6), the fact that $\varepsilon_1 \in [1,2]$, and (4-5) (iii), we find that

$$\begin{aligned}\frac{\log(1/\sigma_1)}{\log(\sigma/\sigma_1)} &= \frac{\log(1/\varepsilon_1) + \big(b\nu(2\theta_0 r)\big)^{1/(n-1)}}{\log\sigma + \log(1/\varepsilon_1) + \big(b\nu(2\theta_0 r)\big)^{1/(n-1)}} \\ &\le \frac{\big(b\nu(2\theta_0 r)\big)^{1/(n-1)}}{\log\sigma + \log(1/2) + \big(b\nu(2\theta_0 r)\big)^{1/(n-1)}} \\ &= \frac{1}{1 + (\log\sigma/2)/\big(b\nu(2\theta_0 r)\big)^{1/(n-1)}} \le 2^{1/n}.\end{aligned} \tag{4-12}$$

Also, since $\varepsilon_1 < 2$, (4-6) and (4-5) (iv) yield that

$$\frac{\sigma}{\sigma_1} > \frac{2\exp\{-\frac{1}{2}\big(b\nu(2\theta_0 r)\big)^{1/n}\}}{\varepsilon_1 \exp\{-\big(b\nu(2\theta_0 r)\big)^{1/(n-1)}\}} > \exp\{\tfrac{1}{2}\big(b\nu(2\theta_0 r)\big)^{1/(n-1)}\},$$

and hence

$$\Big(\log\frac{\sigma}{\sigma_1}\Big)^{n-1} > \frac{b\nu(2\theta_0 r)}{2^{n-1}}. \tag{4-13}$$

Substituting (4-12) and (4-13) into (4-11) we get

$$I \le K_O\omega_{n-1}2^{-n}\Big[\frac{2^{n-1}}{b} + \frac{2c_1}{n}\Big] \le \frac{K_O\omega_{n-1}}{2b} + \frac{\omega_{n-1}K_O c_1}{2^{n-1}n}.$$

The same estimate holds for II. Substituting these and the value of b from (4-3) into (4-10) we obtain

$$M(\Gamma) \le \frac{K_O\omega_{n-1}}{b} + \frac{\omega_{n-1}K_O c_1}{2^{n-2}n} = \frac{c_n\log\theta_0}{2} + \frac{\omega_{n-1}K_O c_1}{2^{n-2}n}.$$

Further by [V1, (10.12)], $M(\Gamma) \ge c_n\log\theta_0$ so that

$$\frac{c_n\log\theta_0}{2} \le \frac{\omega_{n-1}K_O c_1}{2^{n-2}n}.$$

But this contradicts our choice of θ_0 in (4-3). This proves the lemma.

From this lemma, we find that for $j \ne J(r)$, $\mathscr{F}_j \subset B(\theta_0 r)$. If $J(r)$ does not exist, so that $\mathscr{F}_j \subset B(\theta_0 r)$ for all j, we then set $J(r) = q$. Fix $j \ne J$, and $y \in S$. Then $\Lambda_y^j = \{\lambda_1, \dots, \lambda_h\} \subset B(\theta_0 r)$, and since Λ_y^j is a maximal sequence of $f \,|\, B(4\theta_0 r + 1)$ lifts of $\gamma_y^j \,|\, [0,1/2]$ we have , for all $j \ne J$, $y \in S$,

$$h = n(r, a_j) \le n\big(\theta_0 r, \alpha_j(y)\big). \tag{4-14}$$

Now set

$$A_j = S(a_j, \sigma_1) \cap \{ f\big(B_f \cap \bar{B}(8\theta_0 r)\big) \cup f\big(\partial B(\theta_0 r)\big) \} \tag{4-15}$$

where B_f is the branch set, i.e. the set of points where f is not a local homeomorphism. From [MR, 3.1] we note that for all $j = 1, \dots, q$,

$$\mathscr{H}^{n-1}\big(S(a_j, \sigma_1) \cap f\big(B_f \cap \bar{B}(8\theta_0 r)\big)\big) = 0.$$

This along with (4-7) implies that $\mathscr{H}^{n-1}(A_j) = 0$ for all j. Further, we have that $\mathscr{H}^{n-1}\big({\alpha_j}^{-1}(A_j)\big) = 0$ for all j.

Set

$$S' = S \setminus \Big[\textstyle\bigcup_{j=1}^{q} {\alpha_j}^{-1}(A_j)\Big]. \tag{4-16}$$

Comparison of $n\big(\theta_0 r, \alpha_j(y)\big)$ *with* $n\big(2\theta_0 r, \beta_j(y)\big)$. For any $y \in S$, we redefine $\Lambda_y^j = \{\lambda_1, \dots, \lambda_g\}$ to be a maximal sequence of $f \,|\, B(4\theta_0 r + 1)$-liftings of $\gamma_y^j \,|\, [1/2, 1]$, starting at points of $f^{-1}\big(\alpha_j(y)\big) \cap \bar{B}(\theta_0 r)$, where $g = n\big(\theta_0 r, \alpha_j(y)\big)$. Let the set of such sequences be Ω_y^j. For $\Lambda_y^j \in \Omega_y^j$ we set

$$N(\Lambda_y^j) = \operatorname{card}\{\, \nu : |\lambda_\nu| \subset \bar{B}(2\theta_0 r) \,\}$$

and define

$$p_j(y) = \sup_{\Lambda_y^j \in \Omega_y^j} N(\Lambda_y^j). \tag{4-17}$$

Fix an extremal sequence $\hat{\Lambda}_y^j \in \Omega_y^j$; i.e. $N(\hat{\Lambda}_y^j) = p_j(y)$. Then by the definition of a maximal sequence of f-liftings, we have,

$$p_j(y) \le n\big(2\theta_0 r, \beta_j(y)\big). \tag{4-18}$$

We shall integrate $n\big(\theta_0 r, \alpha_j(y)\big) - p_j(y)$ on S and for this we need the following lemma, which is almost entirely an imitation of [R4, 4.1].

Lemma 4-19. *Let* S' *and* p_j *be as in* (4-16) *and* (4-17), *then* p_j *is upper semi-continuous on* S'.

Proof. Let $y_0 \in S'$, then by (4-16) and (4-15), $\alpha_j(y_0) \notin f\big(B_f \cap \bar{B}(8\theta_0 r)\big) \cup f\big(\partial B(\theta_0 r)\big)$. So if $f^{-1}\big(\alpha_j(y_0)\big) \cap \bar{B}(\theta_0 r) = \{x_1, \dots, x_g\}$, with $g = n\big(\theta_0 r, \alpha_j(y_0)\big)$, then $\{x_1, \dots, x_g\} \subset B(\theta_0 r)$. Let $y_1, y_2, \dots$ be a sequence in S' such that $y_h \to y_0$. The lemma asserts that

$$\limsup_{h \to \infty} p_j(y_h) \le p_j(y_0).$$

By choosing a subsequence we may assume that for some integer m, $p_j(y_h) \equiv m$ holds for all $h \geq 1$. Also $n\big(\theta_0 r, \alpha_j(y)\big)$ is upper semi-continuous in y because $n(r,y)$ is. Hence if $g_h = n\big(\theta_0 r, \alpha_j(y_h)\big)$, then $\limsup_{h\to\infty} g_h \leq g$. We choose and fix the following:

(i) Normal neighbourhoods $V_1, \dots, V_g \subset B(\theta_0 r)$ of the points $x_1, \dots, x_g$, respectively, such that $\alpha_j(y_h) \in \bigcap_{\nu=1}^{g} f(V_\nu)$, $h \geq 1$. (This then implies $f^{-1}\big(\alpha_j(y_h)\big) \cap V_\nu \neq \emptyset$ for all ν, so that $g_h \geq g$; i.e. $g_h = g$.)

(ii) For each $h \geq 1$ a maximal sequence $\hat{\Lambda}^j_{y_h} = \{\lambda_{h,1}, \dots, \lambda_{h,g}\} \in \Omega^j_{y_h}$ such that $\lambda_{h,\nu}$ starts at a point $\zeta_{h,\nu}$ in $f^{-1}\big(\alpha_j(y_h)\big) \cap V_\nu$ for $\nu = 1, \dots, g$, and $|\lambda_{h,\nu}| \subset \bar{B}(2\theta_0 r)$ for $\nu = 1, \dots, m$ (since $p_j(y_h) \equiv m$).

We divide the ν's, $1 \leq \nu \leq g$, into two groups. First let $\nu \in \{1, \dots, m\}$ be fixed. We claim that the family $\{\, \lambda_{h,\nu} : h = 1, 2, \dots\}$ is equicontinuous on $1/2 \leq t \leq 1$. Indeed, choose $\varepsilon > 0$. For $t \in [1/2, 1]$ there exists $\delta_t > 0$ such that $U(\xi, f, \varrho)$ is a normal neighbourhood of ξ with $d\big(U(\xi, f, \varrho)\big) < \varepsilon$ for each $\xi \in f^{-1}\big(\gamma^j_{y_0}(t)\big) \cap \bar{B}(2\theta_0 r)$, and

$$\text{(4-20)} \quad \bar{B}(2\theta_0 r) \cap f^{-1}\big(B\big(\gamma^j_{y_0}(t), \varrho\big)\big) \subset \textstyle\bigcup_\xi \big\{U(\xi, f, \varrho) : \xi \in f^{-1}\big(\gamma^j_{y_0}(t)\big) \cap \bar{B}(2\theta_0 r)\big\}$$

whenever $0 < \varrho < \delta_t$. We cover $\gamma^j_{y_0}([1/2,1])$ with a finite number of balls $B(\gamma^j_{y_0}(t), \delta_t/2)$, say $B(\eta_u, \varrho_u), u = 1, \dots, v$. Again by taking a subsequence of the $\{y_h\}$ we have $\gamma^j_{y_h}([1/2,1]) \subset \bigcup_{u=1}^{v} B(\eta_u, \varrho_u)$, and $|\alpha_j(y_h) - \alpha_j(y_0)| \leq \delta = \min_{1\leq u\leq v}\{\varrho_u/8\}$, $|\beta_j(y_h) - \beta_j(y_0)| \leq \delta$ for all $h \geq 1$. Fix $t \in [1/2, 1]$. Since γ is continuous there exists u such that for any $h \geq 1$

$$\gamma^j_{y_h}(t') \in B(\eta_u, 2\varrho_u) \qquad \text{for} \ \ |t' - t| < \delta.$$

For each such h there exists then $\xi \in f^{-1}(\eta_u) \cap \bar{B}(2\theta_0 r)$ such that, by (4-20)

$$|\lambda_{h,\nu}(t')| \subset U(\xi, f, 2\varrho_u) \qquad \text{for} \ \ |t' - t| < \delta.$$

And since $d\big(U(\xi, f, 2\varrho_u)\big) < \varepsilon$ for all $h \geq 1$, the family $\{\lambda_{h,\nu}\}_{h\geq 1}$ is equicontinuous. By Ascoli's theorem we may conclude that $\{\lambda_{h,\nu}\}_{h\geq1}$ converges uniformly to a path λ_ν: $[1/2,1] \to \bar{B}(2\theta_0 r)$. The path λ_ν is a maximal $f \,|\, B(4\theta_0 r + 1)$-lift of $\gamma^j_{y_0} \,|\, [1/2,1]$.

Next fix $\nu \in \{m+1, \dots, g\}$. Let the end-point of $\lambda_{h,\nu}$, in $B(4\theta_0 r + 1)$, occur at $t = t_h < 1$ and set $t_0 = \limsup_{h\to\infty} t_h$. We shall construct a maximal $f \,|\, B(4\theta_0 r + 1)$-lift λ_ν of $\gamma^j_{y_0} \,|\, [1/2,1]$ with end-point t_0 as follows. By taking subsequences of $\{t_h\}$ again, we may assume $t_0 = \lim_{h\to\infty} t_h$. As above we conclude that the paths $\lambda_{h,\nu} \circ G_h$, where G_h maps $[1/2, t_0)$ affinely onto $[1/2, t_h)$, converges uniformly on compact subsets of $[1/2, t_0)$ to a path $\tilde{\lambda}_\nu$: $[1/2, t_0) \to \bar{B}(4\theta_0 r + 1)$ which is then a lift of $\gamma^j_{y_0} \,|\, [1/2, t_0)$. The path has an extension to a path $\bar{\lambda}_\nu$: $[1/2, t_0] \to \bar{B}(4\theta_0 r + 1)$, by [MRV3, 3.12]. If $\Delta \subset [1/2, t_0]$ is the largest interval such that

$1/2 \in \Delta$ and $\bar{\lambda}_\nu \Delta \subset \bar{B}(4\theta_0 r + 1)$, then $\lambda_\nu = \bar{\lambda}_\nu \,|\, \Delta$ is maximal $f \,|\, B(4\theta_0 r + 1)$-lift of $\gamma^j_{y_0} \,|\, [1/2, 1]$, and we have constructed paths $\lambda_1, \dots, \lambda_g$, each of which is a maximal lift of $\gamma^j_{y_0} \,|\, [1/2, 1]$. Next we will show that $\Lambda_{y_0} = \{\lambda_1, \dots, \lambda_g\} \in \Omega_{y_0}$; i.e. Λ_{y_0} is a maximal sequence of $f \,|\, B(4\theta_0 r + 1)$-liftings of $\gamma^j_{y_0} \,|\, [1/2, 1]$, as defined in [R1]. We need only check that

$$\operatorname{card}\{\, \nu : \lambda_\nu(t) = x \,\} \le i(x, f) \qquad \text{for all } t \text{ and } x.$$

Let $A = \{\nu : \lambda_\nu(t) = x\} \ne \emptyset$, and let $U(x, f, \varrho)$ be normal neighbourhood of x. There exists h_0 such that $|\lambda_{h,\nu}| \cap U \ne \emptyset$ for all $h \ge h_0$, $\nu \in A$. Let $h \ge h_0$. We may easily find a point $\eta = \gamma^j_{y_h}(t')$ in $\bigcap_{\nu \in A} \{f(|\lambda_{h,\nu}| \cap U)\}$. Let $\xi_1, \dots, \xi_w$ be the points in $\{\, \lambda_{h,\nu}(t') : \nu \in A \,\} \subset f^{-1}(\eta) \cap U$. Since $\{\lambda_{h,1}, \dots, \lambda_{h,g}\}$ is a maximal sequence, we have for $u = 1, \dots, w$,

$$\theta_u = \operatorname{card}\{\, u : \lambda_{h,\nu}(t') = \xi_u \,\} \le i(\xi_u, f).$$

Further, by the choice of η and since U is a normal neighbourhood of x,

$$\operatorname{card} A = \sum_{u=1}^{w} \theta_u \le \sum_{u=1}^{w} i(\xi_u, f) \le n(U, \eta) = n(U, x) = i(x, f),$$

where the last inequality is true because $f^{-1}\big(f(x)\big) \cap U = \{x\}$. This proves that $\Lambda_{y_0} = \{\lambda_1, \dots, \lambda_g\}$ obtained above is a maximal sequence of $f \,|\, B(4\theta_0 r + 1)$-liftings of $\gamma^j_{y_0} \,|\, [1/2, 1]$, such that $|\lambda_\nu| \subset \bar{B}(2\theta_0 r)$ for $1 \le \nu \le m$. Thus $p_j(y_0) \ge N(\Lambda_{y_0}) = m$. This proves the lemma.

Set

$$q_j(y) = n\big(\theta_0 r, \alpha_j(y)\big) - p_j(y). \tag{4-21}$$

q_j, being the difference of two measurable functions, is measurable relative to S'. With $\hat{\Lambda}^j_y$ such that $p_j(y) = N(\hat{\Lambda}^j_y)$, for $k = 1, 2, \dots$, let

$$\begin{aligned}
E^j_k &= \{\, y \in S' : q_j(y) = k \,\}, \qquad E^{j\,\prime}_k = \{\, y + a_j : y \in E^j_k \,\} \\
\Gamma^j_k &= \{\, \gamma^j_y \,|\, [1/2, 1] : y \in E^j_k \,\} \\
\Delta^j_k &= \{\, \lambda_\nu : \lambda_\nu \in \hat{\Lambda}^j_y, y \in E^j_k, |\lambda_\nu| \not\subseteq \bar{B}(2\theta_0 r) \,\}.
\end{aligned}$$

Then $\mathscr{H}^{n-1}(E^j_k) = \mathscr{H}^{n-1}(E^{j\,\prime}_k)$ and by the definition of E^j_k and the fact that $\mathscr{H}^{n-1}(S \setminus S') = 0$, we have

$$\begin{aligned}
\frac{1}{\omega_{n-1}} \int_S q_j(y)\, d\mathscr{H}^{n-1}(y) &= \frac{1}{\omega_{n-1}} \sum_{k=1}^{\infty} k \mathscr{H}^{n-1}(E^j_k) \\
&= \frac{1}{\omega_{n-1}} \sum_{k=1}^{\infty} k \mathscr{H}^{n-1}(E^{j\,\prime}_k).
\end{aligned} \tag{4-22}$$

We get $\mathscr{H}^{n-1}(E_k^{j'}) = \bigl(\log(\sigma_2/\sigma_1)\bigr)^{n-1} M(\Gamma_k^j)$ using a standard estimate, [V1, 7.7]. Thus (4-22) becomes

$$\frac{1}{\omega_{n-1}} \int_S q_j(y)\, d\mathscr{H}^{n-1}(y) = \frac{1}{\omega_{n-1}} \sum_{k=1}^{\infty} kM(\Gamma_k^j)\bigl(\log(\sigma_2/\sigma_1)\bigr)^{n-1} = \frac{1}{\omega_{n-1}} \bigl(\log(\sigma_2/\sigma_1)\bigr)^{n-1} \sum_{k=1}^{\infty} kM(\Gamma_k^j).$$

Further, Väisälä's inequality [V2, 3.1] gives us $kM(\Gamma_k^j) \le K_I M(\Delta_k^j)$. Also note that since the $\{\Gamma_k^j\}_{j,k}$ are disjoint, so are the $\{\Delta_k^j\}_{j,k}$, and by [V1, 6.7],

$$\sum_{\substack{bj=1\\ j\neq J}}^{q} \sum_{k=1}^{\infty} M(\Delta_k^j) \le M\Bigl(\bigcup_{\substack{j=1\\ j\neq J}}^{q} \bigcup_{k=1}^{\infty} \Delta_k^j\Bigr).$$

Using these two estimates, summing over $j \neq J$ and recalling σ_1 from (4-6) we get

$$\begin{aligned}\sum_{j\neq J} \frac{1}{\omega_{n-1}} \int_S q_j(y)\, d\mathscr{H}^{n-1}(y) &\le \frac{1}{\omega_{n-1}} \bigl(\log(\sigma_2/\sigma_1)\bigr)^{n-1} K_I M\Bigl(\bigcup_{\substack{j=1\\ j\neq J}}^{q} \bigcup_{k=1}^{\infty} \Delta_k^j\Bigr)\\ &\le \frac{1}{\omega_{n-1}} \bigl(\log(\sigma_2/\sigma_1)\bigr)^{n-1} K_I \frac{\omega_{n-1}}{(\log 2)^{n-1}} \qquad (4\text{-}23)\\ &\le \frac{K_I}{(\log 2)^{n-1}} b\nu(2\theta_0 r).\end{aligned}$$

If $y \in S$, then by (4-14), (4-21) and (4-18) we have

$$n(r, a_j) \le q_j(y) + n\bigl(2\theta_0 r, \beta_j(y)\bigr). \tag{4-24}$$

On integrating over S, and summing over $j \neq J$, we obtain using (4-23),

$$\sum_{j\neq J} n(r, a_j) \le \frac{K_I b\nu(2\theta_0 r)}{(\log 2)^{n-1}} + \sum_{j\neq J} \nu\bigl(2\theta_0 r, S(a_j, \sigma_2)\bigr). \tag{4-25}$$

But from (2-3) and (4-8)

$$\nu\bigl(2\theta_0 r, S(a_j, \sigma_2)\bigr) \le \nu(4\theta_0 r) + c_1\bigl(b\nu(2\theta_0 r)\bigr)^{1-1/n}.$$

Using this (4-25) becomes

$$\sum_{j\neq J} n(r, a_j) \le \frac{K_I b}{(\log 2)^{n-1}} \nu(2\theta_0 r) + (q-1)\nu(4\theta_0 r) + (q-1)c_1\bigl(b\nu(2\theta_0 r)\bigr)^{1-1/n}.$$

Finally we use (4-5) (v) in the above inequality to get

$$\sum_{j\neq J} n(r,a_j) \le \frac{K_I b}{(\log 2)^{n-1}}\nu(2\theta_0 r) + (q-1)\nu(4\theta_0 r) + \nu(2\theta_0 r)$$
$$\le \Big[q + \frac{K_I b}{(\log 2)^{n-1}}\Big]\nu(4\theta_0 r).$$

In the situation when $\tau > 1$ this gives us

$$\sum_{j\neq J} n(r,a_j) \le \Big[q + \frac{K_I b}{(\log 2)^{n-1}}\Big]\nu(4\theta_0 r, \tau).$$

Step III: Recall $r_0 = \max(r_1, r_2)$. Fix $r \ge r_0$, and use (2-2) to replace $\nu(4\theta_0 r, \tau)$ by $\nu(8\theta_0 r)$ to get,

$$\sum_{j\neq J} n(r,a_j) \le \Big[q + \frac{K_I b}{(\log 2)^{n-1}}\Big]\Big(\nu(8\theta_0 r) + K_I\Big(\frac{\log\tau}{\log 2}\Big)^{n-1}\Big).$$

The inequality (4-4) (i) reduces this to

$$\sum_{j\neq J} n(r,a_j) \le \Big[q + \frac{2K_I b}{(\log 2)^{n-1}}\Big]\nu(8\theta_0 r),$$

and by (4-4) (ii) we obtain,

$$\sum_{j\neq J} n(r,a_j) \le \Big[q + \frac{4K_I b}{(\log 2)^{n-1}} + 2\Big]A(16\theta_0 r).$$

This proves Theorem 4-1.

Theorem 4-26. *For $n \ge 2$, and $K \ge 1$, let $f\colon \mathbf{R}^n \to \bar{\mathbf{R}}^n$ be a nonconstant K-qm function. Then there exist constants $C_1 = C_1(n,K) > 1$, $\theta_1 = \theta_1(n,K) > 1$ such that for every $a_1,\dots,a_q \in \bar{\mathbf{R}}^n$, $q > 1$, there exists a set $E \subset [1,\infty)$ with $\int_E d\lambda/\lambda = \infty$ such that*

$$\limsup_{\substack{r\to\infty \\ r\in E}} \sum_{j=1}^{q} \Big[\frac{n(r,a_j)}{A(\theta_1 r)} - 1\Big]_+ \le C_1. \tag{4-27}$$

Proof of Theorem 4-26. We first use Theorem 3-1 with some fixed value of C, say $C = 2$, and obtain a corresponding θ and a set $E \subset [1,\infty)$ with $\int_E d\lambda/\lambda = \infty$, such that for $j = 1,\dots,q$, $r \in E$,

$$n(r,a_j) \le 2A(\theta r). \tag{4-28}$$

We will then show that (4-27) holds with

$$\theta_1 = \max(16\theta_0, \theta), \qquad C_1 = 4 + \frac{4K_I b}{(\log 2)^{n-1}}$$

where b has been defined in (4-3). As in Theorem 4-1, we assume that $a_1, \ldots, a_q \in B(\tau/2)$ and $\sigma > 0$ such that $B_{\sigma\tau}(a_j) \subset B(\tau/2)$ and $\bar{B}_{\sigma\tau}(a_j)$ are disjoint.

Now apply Theorem 4-1 and obtain $r_0 = r_0(\sigma, \tau, q, f) > 0$. Fix $r \in E$ such that $r \geq r_0$. If $\big((n(r, a_j)/A(\theta_1 r)) - 1\big) \leq 0$ for $(q-1)$ values of j, then by (4-28) there is nothing to prove. So let $Q = \big\{1 \leq j \leq q : \big((n(r, a_j)/A(\theta_1 r)) - 1\big) > 0\big\}$ for all $j \in Q$. We assume $\operatorname{card} Q = q' \geq 2$.

Again we apply Theorem 4-1, to the same function f, but using the set $\{a_j : j \in Q\} = \{a'_j\}$. Note that the same σ and τ, as for the $\{a_j\}$, work for $\{a'_j\}$. Theorem 4-1 yields $r'_0 = r'_0(\sigma, \tau, q', f)$. From (4-4) and (4-5) (v) we see that we may choose $r'_0(\sigma, \tau, q', f) = r_0(\sigma, \tau, q, f)$; i.e. $r'_0 = r_0$. So we have for $r \in E$, $r \geq r_0 = r'_0$, by (4-2),

$$\sum_{\substack{j \in Q \\ j \neq J}} n(r, a_j) \leq \Big[q' + \frac{4K_I b}{(\log 2)^{n-1}} + 2\Big] A(16\theta_0 r) \leq \Big[q' + \frac{4K_I b}{(\log 2)^{n-1}} + 2\Big] A(\theta_1 r);$$

i.e.,

$$\sum_{\substack{j \in Q \\ j \neq J}} \Big[\frac{n(r, a_j)}{A(\theta_1 r)} - 1\Big] \leq \Big[3 + \frac{4K_I b}{(\log 2)^{n-1}}\Big].$$

For $j = J$, since $r \in E$, we have from (4-28) that

$$n(r, a_J) \leq 2A(\theta r) \leq 2A(\theta_1 r).$$

Hence

$$\sum_{j \in Q} \Big[\frac{n(r, a_j)}{A(\theta_1 r)} - 1\Big] \leq \Big[4 + \frac{4K_I b}{(\log 2)^{n-1}}\Big] = C_1.$$

And by the definition of Q,

$$\sum_{j=1}^{q} \Big[\frac{n(r, a_j)}{A(\theta_1 r)} - 1\Big]_+ \leq C_1,$$

where $r \in E$, $r \geq r_0$. The theorem is proved.

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