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ON BOUNDARY CORRESPONDENCE UNDER QUASICONFORMAL MAPPINGS

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Abstract. We study boundary properties of quasiconformal self-mappings depending on complex dilatations. We give some new conditions for the corresponding quasisymmetric function to be asymptotically symmetric and obtain an explicit asymptotical representation for the distortion ratio of boundary correspondence when the complex dilatation has directional limits.

1. Introduction

Let $\overline{\mathbf{C}}$ be the extended complex plane. We shall denote the unit disk $|z|<1$ by D, the unit sphere $|z|=1$ by S^1 and the upper half-plane $\operatorname{Im} z>0$ by $\mathbf{H}$.

An orientation preserving $\mathscr{ACL}$ homeomorphism f is called *quasiconformal* if it satisfies the *Beltrami equation*

$$f_{\bar z}=\mu(z)f_z \qquad \text{a.e.} \tag{1.1}$$

for some measurable complex function μ called the *Beltrami differential* or *complex dilatation* with

$$\operatorname*{ess\,sup}_G |\mu(z)| = \|\mu\|_\infty \le k < 1. \tag{1.2}$$

If $Q=(1+k)/(1-k)$ then f is called also Q-quasiconformal.

An increasing self-homeomorphism f of the real axis $\mathbf{R}$ is called *quasisymmetric* if it can be extended to a quasiconformal mapping of the upper half-plane $\mathbf{H}$ that fixes the point at infinity.

It is a well-known fundamental result that the boundary values of a Q-quasiconformal self-mapping f of the upper half-plane preserving the point at infinity satisfy the double sharp inequality (see [20, p. 81])

$$\frac{1}{\lambda(Q)} \le \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \le \lambda(Q) \tag{1.3}$$

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for all real x and $t \neq 0$ with

$$\lambda(Q) = \frac{1}{\left(m^{-1}\left(\pi Q/2\right)\right)^2} - 1$$

and $m(r)$ from Grötzsch's module theorem.

In 1956 Beurling and Ahlfors [4] proved the converse having shown that f is quasisymmetric if for some constant M, $1 \leq M < \infty$, *the* M*-condition*

$$\frac{1}{M} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M \tag{1.4}$$

is satisfied for every symmetric triple $x - t$, x and $x + t$ in $\mathbf{R}$.

Next in 1967 Carleson [6] considered quasiconformal self-mappings of $\mathbf{H}$ which "are conformal at the boundary" in the sense that

$$k(t) = \operatorname*{ess\,sup}_{0<\operatorname{Im} z \leq t} |\mu(z)| \to 0, \qquad t \to 0 \tag{1.5}$$

and showed that the assumption

$$\int_0 k(t)^\alpha \frac{dt}{t} < \infty \tag{1.6}$$

for $\alpha = 2$ implies an absolutely continuous boundary correspondence $f(x)$, while for $\alpha = 1$, $f(x)$ is continuously differentiable. He gave also counterexamples in the case when (1.6) does not hold for $\alpha = 1$ or $\alpha = 2$, respectively. Then Anderson, Becker and Lesley [2] investigated the same problems for quasiconformal self-mappings of the disk D by methods of the theory of conformal mappings with asymptotically conformal extension [3]. They have obtained also estimations of the moduli of continuity of $f'(e^{it})$ for $\alpha = 1$ and estimations of the mean oscillation of $\ln|f'(e^{it})| \in \mathrm{VMO}(S^1)$ for $\alpha = 2$.

This paper is concerned with an investigation of symmetry and regularity problems at a given point and on subsets of the complex plane for quasiconformal mappings. It contains some new results related to asymptotical behavior of quasisymmetric homeomorphisms. It is well-known by a principal result of Gehring and Lehto [13] that a general quasiconformal mapping f, being Hölder continuous, is differentiable only almost everywhere and it is necessary to require some additional restrictions on the complex dilatation $\mu(z)$ in order to make more precise statements about the pointwise behaviour of the mapping. We shall choose the upper half-plane as a main canonical domain and the limit

$$\lim_{t\to 0} \frac{1}{t^2} \iint\limits_{|z-x|\leq t, z\in\mathbf{H}} |\mu(z) - \nu_x(z)|^\alpha \, dm_z = 0, \qquad \alpha > 0 \tag{1.7}$$

as a condition for measuring the deviation for the complex dilatation $\mu(z)$. Here $\nu_x(z)$, $x \in \mathscr{M} \subseteq \mathbf{R}$ is an appropriate one-parameter family of the Beltrami differentials. The assumptions we shall require are motivated by problems which arise in the theory of convergence and compactness for quasiconformal mappings.

We first require (1.7) to hold for $\nu_x(z) = \nu_x$, $x \in \mathbf{R}$, that is, $\nu_x(z)$ does not depend on z and prove a result which implies a local asymptotical symmetry at the point x for the corresponding quasisymmetric function

$$\lim_{t\to 0} \frac{f(x+t)-f(x)}{f(x)-f(x-t)} = 1. \tag{1.8}$$

The condition we impose on $\mu(z)$ does not imply the differentiability of f at the point x even if the complex dilatation is continuous at this point.

Then we focus our attention to the study of a boundary correspondence under the similar, but slightly less restrictive, requirement on $\mu(z)$ that (1.7) holds for a function $\nu_x(z)$ depending on $\arg(z-x)$ only for any fixed $x \in \mathscr{M} \subseteq \mathbf{R}$. In this case we prove the more general Theorem 2 about regularity and symmetry under quasiconformal mappings. Note that a complex dilatation $\mu(z)$ having for fixed $x \in \mathbf{R}$ and almost all θ, $0 \le \theta \le \pi$, finite limits in directions

$$\lim_{t\to 0} \mu(x+te^{i\theta}) = \nu_x(e^{i\theta}) \tag{1.9}$$

satisfies the above restriction.

Finally, we apply special cases of the previous results to the study of the boundary behaviour of quasiconformal self-homeomorphisms of the unit disk D.

Different analytic and geometric properties of quasisymmetric homeomorphisms have been investigated, for instance, by Gardiner and Sullivan [9], Gehring [10], Douady and Earle [7], Fehlmann [8], Hayman [17] and Tukia [22]. Note also that quasisymmetric homeomorphisms of an interval into $\mathbf{R}$ play an important role in the theory of Riemann surfaces and of real one dimensional smooth dynamical systems.

2. Main results

We begin with a result on asymptotical symmetry of quasisymmetric self-homeomorphisms of the real axis $\mathbf{R}$.

Theorem 1. *Let f be a Q-quasiconformal self-mapping of the upper half-plane $\mathbf{H}$ with $f(\infty) = \infty$ and complex dilatation μ. If there exists a complex number ν_x such that*

$$\lim_{t\to 0} \frac{1}{t^2} \iint\limits_{|z-x|\le t, z\in\mathbf{H}} |\mu(z)-\nu_x|^\alpha \, dm_z = 0 \tag{2.1}$$

for some $\alpha > 0$ *and fixed* $x \in \mathbf{R}$, *then*

$$\lim_{\substack{t\to 0\\ t\in\mathbf{R}\setminus\{0\}}} \frac{f(x+t)-f(x)}{f(x)-f(x-t)} = 1. \tag{2.2}$$

Moreover, if the limit (2.1) *is uniform with respect to the parameter* $x \in \mathscr{M} \subset \mathbf{R}$ *then* (2.2) *is also uniform in* $\mathscr{M}$.

In what follows $\mathfrak{F}_Q$ is the class of all Q-quasiconformal self-mappings f of the extended complex plane $\overline{\mathbf{C}}$ normalized with the conditions $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$.

Proof. Suppose that f is a Q-quasiconformal mapping of $\mathbf{H}$ onto itself, normalized so that $f(\infty) = \infty$. Then we can extend f by reflection in the half-plane $\operatorname{Im} z < 0$ to obtain a Q-quasiconformal mapping of the complex plane $\overline{\mathbf{C}}$ with $f(\infty) = \infty$. The complex dilatation $\mu(z) = f_{\bar z}/f_z$ will satisfy the symmetry condition $\mu(\bar z) = \overline{\mu(z)}$ almost everywhere in $\mathbf{C}$.

Next, fix $x \in \mathbf{R}$ and introduce the family of quasiconformal mappings $\Phi(z,x,t) = g(tz;x)/g(t;x)$, where $g(z;x) = f(x+z) - f(x)$. It is easy to verify that $\mu(x+tz)$ represents the complex dilatation for the mapping $\Phi(z,x,t) \in \mathfrak{F}_Q$. On the other hand, the assumption (2.1) implies that $\mu(x+tz) \xrightarrow{\text{mes}} \nu_x$ as $t \to 0$, $t > 0$, $z \in \mathbf{H}$. It is well-known that the last assertion implies the existence of a sequence $\mu(x+t_n z)$ which converges to ν_x almost everywhere.

Now we shall show that

$$\lim_{t\to 0} \Phi(z,x,t) = \omega_x(z) \tag{2.3}$$

for all $z \in \mathbf{C}$, where

$$\omega_x(z) = \begin{cases} \dfrac{z+\nu_x \bar z}{1+\nu_x} & \text{for } \operatorname{Im} z \ge 0,\\[2mm] \dfrac{z+\overline{\nu_x}\bar z}{1+\overline{\nu_x}} & \text{for } \operatorname{Im} z \le 0. \end{cases}$$

Suppose that (2.3) fails. Then there exists an $\varepsilon > 0$ and a sequence $t_n \to 0$ as $n \to \infty$ such that for some $z_0 \in \mathbf{C}$

$$|\Phi(z_0,x,t_n) - \omega_x(z_0)| \ge \varepsilon.$$

Without loss of generality we can assume also that the corresponding sequence of quasiconformal mappings $\Phi(z,x,t_n) \in \mathfrak{F}_Q$ converges locally uniformly to some function $\Phi(z,x) \in \mathfrak{F}_Q$ and $\mu(x+t_n z)$, $z \in \mathbf{H}$, converges to ν_x almost everywhere. Now by the Bers–Bojarski convergence theorem (see [20, p. 187]) the complex dilatation of the limit function $\Phi(z,x) \in \mathfrak{F}_Q$ will agree with ν_x for almost all

$z \in \{z : \operatorname{Im} z \geq 0\}$ and, by symmetry, with $\overline{\nu_x}$ for almost all $z \in \{z : \operatorname{Im} z \leq 0\}$. Hence $\Phi(z,x) = \omega_x(z)$. This contradicts the above assumption.

Setting $z = -1$ in (2.3) we complete the proof of the local variant of Theorem 1 emphasizing by the same token the role of fundamental convergence theorems in our considerations.

It is easy to see that Theorem 1 is the partial case of a more general Theorem 2 below. However we decided to extract it as an independent statement to call attention to the symmetry notion for quasisymmetric functions $f\colon \mathbf{R} \to \mathbf{R}$, being expressed by the uniform condition (2.2) in connection with new remarkable applications of the concept given by Gardiner and Sullivan [9].

Going over to the analysis of the above result we first note that if f is differentiable at $x \in \mathscr{M}$ then evidently (2.2) holds. On the other hand, the following example

$$f(z) = z(1 - \log|z|), \qquad f(0) = 0$$

shows that even the continuity of the complex dilatation in a neighbourhood of a prescribed point does not imply a differentiability of the corresponding quasiconformal mapping at this point. In connection with it the following statement may have independent interest.

Corollary 1. *Let f be a quasiconformal self-mapping of the upper half-plane $\mathbf{H}$ with complex dilatation μ that can be extended to a function being continuous uniformly in $\mathscr{M} \subset \mathbf{R}$. Then (2.2) holds uniformly in $\mathscr{M}$.*

Indeed, the convergence of the complex dilatation $\mu(z)$ to $\mu(x) = \nu_x$ as $z \to x$ uniformly with respect to $x \in \mathscr{M}$ implies that the limit (2.1) is also uniform in $\mathscr{M}$ for any $\alpha > 0$.

Denote by $\delta(t,x)$ the essential module of continuity for a complex dilatation $\mu(z)$ extended by $\mu(x)$ at the prescribed point $x \in \mathbf{R}$ that is,

$$\delta(t,x) = \operatorname*{ess\,sup}_{\substack{|z-x|\leq t\\ \operatorname{Im} z>0}} |\mu(z) - \mu(x)|. \tag{2.4}$$

Corollary 2. *If for some $\alpha > 0$ the Dini condition*

$$\lim_{\varepsilon\to 0} \int_0^{\varepsilon} \frac{\delta^{\alpha}(t,x)}{t}\, dt = 0 \tag{2.5}$$

holds uniformly with respect to $x \in \mathscr{M}$ then (2.2) holds also uniformly in $x \in \mathscr{M}$.

The proof of Corollary 2 follows immediately from the evident inequalities

$$\begin{aligned}\frac{1}{t^2} \iint\limits_{|z-x|\leq t, z\in\mathbf{H}} |\mu(z) - \mu(x)|^{\alpha}\, dm_z &\leq \iint\limits_{|z-x|\leq t, z\in\mathbf{C}} \frac{|\mu(z) - \mu(x)|^{\alpha}}{|z-x|^2}\, dm_z \\ &\leq 2\pi \int_0 \frac{\delta^{\alpha}(t,x)}{t}\, dt.\end{aligned} \tag{2.6}$$

Note that the intermediate integral has been introduced by Teichmüller, Wittich, Belinskii and Lehto (see [20, p. 232]) to prove that its convergence for $\alpha \le 1$ implies the differentiability of f at the point x. It can be shown that the uniform convergence of the integral on $\mathscr{M} \subset \mathbf{R}$ for $\alpha \le 1$ implies the uniform differentiability of f in $\mathscr{M}$. Moreover, assumption (2.1) does not imply in general the Teichmüller–Wittich–Belinskii–Lehto condition and the last one does not imply the Dini condition (see [15], [16]).

Before giving the following principal statements we need two preliminary results. The first one deals with an explicit representation of normalized quasiconformal self-mappings of the complex plane $\mathbf{C}$ for a special class of complex dilatations which is due to Schatz [21].

Proposition 1. *Let ω be a quasiconformal self-mapping of the complex plane normalized by the conditions $\omega(0) = 0$, $\omega(1) = 1$, $\omega(\infty) = \infty$ with complex dilatation ν depending on $\arg z$ only. Then*

$$\omega(z) = \left\{ |z| \exp\left(i \int_0^{\arg z} \frac{e^{2i\theta} - \nu(e^{i\theta})}{e^{2i\theta} + \nu(e^{i\theta})}\, d\theta \right) \right\}^{1/b}, \tag{2.7}$$

where

$$b = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{2i\theta} - \nu(e^{i\theta})}{e^{2i\theta} + \nu(e^{i\theta})}\, d\theta. \tag{2.8}$$

The proof follows by straightforward verification.

Remark. The radial lines can be transformed by ω to spirals if and only if $\operatorname{Im} b \neq 0$. If $\operatorname{Im} b = 0$ then all radial lines are translated to radial lines. Moreover, the real axis and upper half-plane are preserved if and only if

$$\operatorname{Re} \int_0^{\pi} \frac{e^{2i\theta} - \nu(e^{i\theta})}{e^{2i\theta} + \nu(e^{i\theta})}\, d\theta = \operatorname{Re} \int_{\pi}^{2\pi} \frac{e^{2i\theta} - \nu(e^{i\theta})}{e^{2i\theta} + \nu(e^{i\theta})}\, d\theta. \tag{2.9}$$

Note that $\nu(e^{-i\theta}) = \overline{\nu(e^{i\theta})}$ implies (2.9) and therefore $b = \operatorname{Re} c$, where

$$c = \frac{1}{\pi} \int_0^{\pi} \frac{e^{2i\theta} - \nu(e^{i\theta})}{e^{2i\theta} + \nu(e^{i\theta})}\, d\theta. \tag{2.10}$$

Example. If $\nu = \kappa(z/\bar{z})$, $\kappa \in \mathbf{C}$, $|\kappa| < 1$, then ω is given by

$$\omega(z) = \frac{z}{|z|} |z|^{c_0}, \tag{2.11}$$

where

$$c_0 = \frac{1+\kappa}{1-\kappa} \tag{2.12}$$

is the corresponding complex parameter with positive real part.

If we set $c_0 = \alpha - i$, $\alpha > 0$, we obtain a quasiconformal mapping of $\mathbf{C}$

$$h(re^{i\theta}) = r^{\alpha} e^{i(\theta - \ln r)} \tag{2.13}$$

mapping the lines $\arg z = \text{const}$ onto spirals. Note that such mappings have been used by Gehring [12] to solve the Bers problem about the structure of the universal Teichmüller space.

The following lemma connects some convergence and regularity problems for quasiconformal mappings and may have an independent interest.

Let $j \in J$ be an abstract parameter. Denote by $\omega_j(z)$ quasiconformal mappings (2.7) with complex dilatations ν_j depending on $\arg z$ only.

Lemma. *Let $f_j\colon \mathbf{C} \to \mathbf{C}$, $f_j(0) = 0$, $j \in J$, be a family of Q-quasiconformal mappings with complex dilatations μ_j such that for some $\alpha > 0$*

$$\lim_{t\to 0} \frac{1}{t^2} \iint_{|z|\le t} |\mu_j(z) - \nu_j(z)|^{\alpha}\, dm_z = 0 \tag{2.14}$$

uniformly with respect to $j \in J$. Then

$$\lim_{\substack{t\to 0\\ t\in \mathbf{R}\setminus\{0\}}} \frac{f_j(tz)}{f_j(t)} = \omega_j(z), \qquad \textit{for all } z \in \mathbf{C}, \tag{2.15}$$

uniformly in $j \in J$.

The proof of the lemma will be a main objective of Section 3.

Theorem 2. *Let f be a Q-quasiconformal self-mapping of the upper half-plane $\mathbf{H}$ with $f(\infty) = \infty$ and complex dilatation μ. If there exists a complex-valued function $\nu_x(\eta)$, $x \in \mathscr{M} \subseteq \mathbf{R}$, $\eta \in S^1$, such that for some $\alpha > 0$*

$$\lim_{t\to 0} \frac{1}{t^2} \iint_{|z-x|\le t,\, z\in\mathbf{H}} |\mu(z) - \nu_x(e^{i\theta})|^{\alpha}\, dm_z = 0 \tag{2.16}$$

where $\theta = \arg(z - x)$, uniformly with respect to $x \in \mathscr{M}$, then

$$\lim_{\substack{t\to 0\\ t\in \mathbf{R}\setminus\{0\}}} \frac{f(x+\tau t) - f(x)}{f(x+t) - f(x)} = \omega_x(\tau), \qquad \textit{for all } \tau \in \mathbf{R} \tag{2.17}$$

uniformly with respect to $x \in \mathscr{M}$. Here $\omega_x(z)$ is given by (2.7) with

$$\nu(e^{i\theta}) = \begin{cases} \nu_x(e^{i\theta}) & \text{for } 0 < \theta < \pi, \\ \overline{\nu_x(e^{-i\theta})} & \text{for } -\pi < \theta < 0. \end{cases}$$

Proof. Suppose that f is a Q-quasiconformal mapping of $\mathbf{H}$ onto itself normalized so that $f(\infty) = \infty$. We shall assume f to be extended in the complex plane by reflection.

Now set $J = \mathscr{M} \subseteq \mathbf{R}$, $j = x \in \mathscr{M}$, $f_j(z) = f(x+z) - f(x)$, $\omega_j(z) = \omega_x(z)$. Applying the lemma to the family $f_j(z)$ yields the theorem.

Corollary 3. *Under the hypothesis of Theorem* 2

$$\lim_{\substack{t\to 0\\ t>0}} \frac{f(x+t)-f(x)}{f(x)-f(x-t)} = e^{\pi \operatorname{Im} c/\operatorname{Re} c}, \tag{2.18}$$

$$\lim_{\substack{t\to 0\\ t\in\mathbf{R}\setminus\{0\}}} \frac{f(x+\tau t)-f(x)}{f(x+t)-f(x)} = \tau^{1/\operatorname{Re} c}, \qquad \textit{for all } \tau>0 \tag{2.19}$$

uniformly with respect to $x\in\mathscr{M}\subseteq\mathbf{R}$.

Let $\omega_x(z)$ is defined by (2.7). Then by straightforward computation it follows that $\omega_x(-1) = -\exp\{-\pi \operatorname{Im} c/\operatorname{Re} c\}$ and $\omega_x(\tau) = \tau^{1/\operatorname{Re} c}$ for $\tau>0$.

Corollary 4. *Under the hypothesis of Theorem* 2 *the property of asymptotical symmetry* (2.2) *holds if and only if* $\operatorname{Im} c = 0$.

Corollary 5. *Under the hypothesis of Theorem* 1 *we have* $c=1$ *and therefore*

$$\lim_{\substack{t\to 0\\ t\in\mathbf{R}\setminus\{0\}}} \frac{f(x+\tau t)-f(x)}{f(x+t)-f(x)} = \tau, \qquad \textit{for all } \tau>0 \tag{2.20}$$

uniformly with respect to $x\in\mathscr{M}\subseteq\mathbf{R}$.

Corollary 6. *Let* f *be a* Q*-quasiconformal self-mapping of the upper half-plane* $\mathbf{H}$ *with complex dilatation* μ *such that for fixed* $x\in\mathscr{M}$ *and almost all* θ, $0\le\theta\le\pi$,

$$\lim_{t\to 0}\mu(x+te^{i\theta}) = \nu_x(e^{i\theta}) \tag{2.21}$$

uniformly with respect to $x\in\mathscr{M}$. *Then* (2.17)–(2.20) *hold also uniformly in* $\mathscr{M}$.

It is easy to show that the uniform assumption (2.21) implies the uniform condition (2.16).

We complete this section with a result on boundary correspondence under quasiconformal mappings of the unit disk onto itself. In analogy with the half-plane model, an orientation-preserving self-homeomorphism f of the unit circle S^1 is called *quasisymmetric* if it can be extended to a quasiconformal mapping of the unit disk D.

Theorem 3. *Let* f *be a quasiconformal self-mapping of the unit disk* D *with complex dilatation* κ. *If there exists a complex number* σ_η *such that*

$$\lim_{t\to 0}\frac{1}{t^2}\iint\limits_{|\zeta-\eta|\le t,\zeta\in D} |\kappa(\zeta)-\sigma_\eta|^\alpha\, dm_\zeta = 0 \tag{2.22}$$

for some $\alpha > 0$ *and fixed* $\eta = e^{i\varphi} \in S^1$, *then the quasisymmetric homeomorphism* $f(e^{i\theta}) = e^{i\Omega(\theta)}$ *satisfies the following condition*

$$\lim_{\tau\to 0} \frac{\Omega(\varphi+\tau)-\Omega(\varphi)}{\Omega(\varphi)-\Omega(\varphi-\tau)} = 1. \tag{2.23}$$

Moreover, if the limit (2.22) *is uniform with respect to a parameter* $\eta \in \mathscr{M} \subseteq S^1$ *then* (2.23) *is also uniform in* $\mathscr{M}$.

Proof. Let f be a quasiconformal self-mapping of the unit disk D. Then

$$h(z) = -i \ln f(e^{iz}) \tag{2.24}$$

is the corresponding quasiconformal self-mapping of the upper half-plane $\mathbf{H}$ for which all hypothesis of Theorem 1 are fulfilled. Indeed, writing $f = \mathscr{A} \circ h \circ \mathscr{A}^{-1}$, where $\mathscr{A}(z) = e^{iz}$, we deduce that

$$\kappa(\zeta) = \Big(\mu \frac{\overline{\mathscr{A}_z}}{\mathscr{A}_z}\Big) \circ \mathscr{A}^{-1}(\zeta).$$

Thus

$$\mu(z) = -\frac{\overline{e^{iz}}}{e^{iz}} \kappa(e^{iz})$$

and therefore

$$\frac{1}{t^2} \iint\limits_{|z-x|\le t, z\in\mathbf{H}} |\mu(z)-\nu_x|^\alpha \, dm_z \sim \frac{1}{t^2} \iint\limits_{|\zeta-\eta|\le t, \zeta\in D} |\kappa(\zeta)-\sigma_\eta|^\alpha dm_\zeta$$

as $t \to 0$. Now assertion (2.23) follows immediately from Theorem 1 being applied to the function $h(z)$.

3. Proof of the lemma

The space $\mathfrak{F}_Q$ of normalized quasiconformal mappings of the complex plane $\mathbf{C}$ with the topology of the locally uniform convergence is metrizable and sequentially compact (see, for instance, [20, p. 71]).

Any metric ϱ defined on $\mathfrak{F}_Q$ is called a generating metric if the convergence $\varrho(f_n, f) \to 0$ is equivalent to the locally uniform convergence $f_n \to f$. One of such metrics is (see, e.g. [18])

$$\varrho(f,g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\varrho_m(f,g)}{1+\varrho_m(f,g)} \tag{3.1}$$

where

$$\varrho_m(f,g) = \max_{|z|\le m} |f(z)-g(z)|. \tag{3.2}$$

Let $\mathfrak{M}_Q$ be the space of all complex dilatations μ for mappings $f \in \mathfrak{F}_Q$. Any metric r defined on $\mathfrak{M}_Q$ is called majorizing if the convergence $r(\mu_n, \nu_n) \to 0$ implies $\varrho(f_n, g_n) \to 0$ for any generating metric ϱ on $\mathfrak{F}_Q$ and for the corresponding mappings f_n and $g_n \in \mathfrak{F}_Q$.

Proposition 2. *Let ϱ and r be generating and majorizing metrics on $\mathfrak{F}_Q$ and $\mathfrak{M}_Q$, respectively. Then $r(\mu_{t,j}, \mu_j) \to 0$ as $t \to 0$ uniformly with respect to an abstract parameter $j \in J$ implies the uniform convergence $\varrho(f_{t,j}, f_j) \to 0$ in $j \in J$.*

Indeed, suppose that there exist an $\varepsilon > 0$ and sequences $t_n \to 0$, $j_n \in J$, such that $\varrho(g_n, h_n) \geq \varepsilon$, where $g_n = f_{t_n, j_n}$, $h_n = f_{j_n}$, $n = 1, 2, \ldots$. However $r(\mu_n, \nu_n) \to 0$ and therefore $\varrho(g_n, h_n) \to 0$. The last statement contradicts the above assumption.

Proposition 3. *For some $\alpha > 0$ let*

$$r(\mu, \nu) = \sum_{m=1}^{\infty} 2^{-m} \frac{r_m(\mu, \nu)}{1 + r_m(\mu, \nu)} \tag{3.3}$$

where

$$r_m(\mu, \nu) = \iint\limits_{|z| \leq m} |\mu(z) - \nu(z)|^{\alpha} \, dm_z. \tag{3.4}$$

Then $r(\mu, \nu)$ is a majorizing metric on $\mathfrak{M}_Q$.

Proof. Assume that $r(\mu_n, \nu_n) \to 0$ but $\varrho(f_n, g_n) \geq \varepsilon > 0$ for some generating metric ϱ and some sequences f_n and $g_n \in \mathfrak{F}_Q$. We may suppose also that $f_n \to f \in \mathfrak{F}_Q$, $g_n \to g \in \mathfrak{F}_Q$.

Next we show that $f \equiv g$. By the Lebesgue convergence theorem $r(\mu_n, \nu_n) \to 0$ is equivalent to $\mu_n - \nu_n \overset{\text{mes}}{\longrightarrow} 0$ on $\mathfrak{M}_Q$. Hence

$$\phi_n(z) = \frac{\mu_n(z) - \nu_n(z)}{1 - \mu_n(z)\overline{\nu_n(z)}} \frac{\overline{(g_n)_z}}{\overline{(g_n)_z}} \overset{\text{mes}}{\longrightarrow} 0.$$

To continue the proof we need the following principal property of normalized Q-quasiconformal mappings related with the area distortion. Let $f \in \mathfrak{F}_Q$ then for each measurable set $E \subset D_R$

$$\text{mes}\big(f(E)\big) \leq c(Q, R)\big(\text{mes}(E)\big)^{\delta(Q)}$$

where $D_R = \{z : |z| < R\}$ and the constants c and δ depend only on Q and R and Q, respectively (see, e.g. [5], [11], [14], [19]).

The above result implies that

$$\kappa_n = \phi_n \circ g_n^{-1} \overset{\text{mes}}{\longrightarrow} 0$$

where the κ_n represent the complex dilatations of the mappings $h_n = f_n \circ g_n^{-1} \in \mathfrak{F}_{Q^2}$. Thus $h_n \to h = f \circ g^{-1}$ and simultaneously $\kappa_n \overset{\text{mes}}{\longrightarrow} 0$. By the Bers–Bojarski convergence theorem (see [20, p. 187]) we deduce that $g = f$. Therefore

$$\varrho(f_n, g_n) \leq \varrho(f_n, f) + \varrho(g_n, f) \to 0$$

as $n \to \infty$. The last contradicts the above assumption.

Proof of the lemma. Let $f_j\colon \mathbf{C} \to \mathbf{C}$, $f_j(0) = 0$, $j \in J$, be a family of Q-quasiconformal mappings with complex dilatations μ_j. It is easy to verify that $\mu_{t,j}(z) = \mu_j(tz)$ represent the complex dilatations for the quasiconformal mappings $g_{t,j}(z) = f_j(tz)/f_j(t) \in \mathfrak{F}_Q$, $t \in \mathbf{R} \setminus \{0\}$. Fix any $\alpha > 0$ and set $t = \tau m$, $z = \tau\zeta$, in (2.14). Then we deduce that

$$\lim_{\tau \to 0} \iint\limits_{|\zeta| \le m} |\mu_{\tau,j}(\zeta) - \nu_j(\zeta)|^\alpha \, dm_\zeta = 0$$

uniformly in $j \in J$ for all $m = 1, 2, \dots$. Now by Propositions 2 and 3 $g_{t,j}(z) \to g_j(z) \in \mathfrak{F}_Q$ locally uniformly with respect to $z \in \mathbf{C}$ uniformly in $j \in J$. Here $g_j(z)$ has the complex dilatation which agrees with $\nu_j(\zeta)$ almost everywhere. By the uniqueness theorem for the Beltrami equation $g_j(z) = \omega_j(z)$. The lemma is proved.

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