Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 21, 2001, 249–266

GEOMETRIC PROPERTIES OF SOLUTIONS TO THE ANISOTROPIC p-LAPLACE EQUATION IN DIMENSION TWO

G. Alessandrini and M. Sigalotti

Università degli Studi di Trieste, Dipartimento di Scienze Matematiche
P. le Europa 1, I-34100 Trieste, Italy
alessang@univ.trieste.it; sigalott@mathsun1.univ.trieste.it

Abstract. We consider solutions to the equation $\operatorname{div}(|A\nabla u\cdot\nabla u|^{(p-2)/2}A\nabla u)=0$ with $1<p<+\infty$, $A=A(x)$ a uniformly elliptic and Lipschitz continuous symmetric matrix, in dimension two. We study the properties of critical points and level lines of such solutions and we apply our results to obtain generalizations of a strong comparison principle due to Manfredi and of a univalence theorem by Radó.

1. Introduction

In this paper we consider solutions $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ to the following degenerate elliptic equation which we shall call the *anisotropic p-Laplace equation*

$$\operatorname{div}(|A\nabla u\cdot\nabla u|^{(p-2)/2}A\nabla u)=0 \qquad \text{in } \Omega, \tag{1.1}$$

where Ω is a two-dimensional domain, p satisfies $1<p<\infty$, and $A=A(x)$ is a symmetric matrix satisfying hypotheses of uniform ellipticity and of Lipschitz continuity.

Equation (1.1) can be viewed as the Euler equation for the variational integral

$$J(u)=\int_\Omega |A\nabla u\cdot\nabla u|^{p/2}\,dx$$

and its interest arises from various applied contexts related to composite materials (such as nonlinear dielectric composites [BS, Section 25], [LK, Section 3.2]), whose nonlinear behavior is modeled by the so-called *power-law*.

Our aim is to study the properties of level lines and of critical points of solutions to (1.1) and to derive some relevant consequences of such properties.

If $p=2$, then equation (1.1) becomes linear:

$$\operatorname{div}(A\nabla u)=0 \qquad \text{in } \Omega. \tag{1.2}$$

2000 Mathematics Subject Classification: Primary 35J65; Secondary 35J70, 30C62, 30G20.
Work supported by Fondi MURST.

In such a case many things are known about the local behavior of solutions and about the structure of level lines and critical points. First, the Hartman and Wintner theorem [HW] tells us that for every $x^0 \in \Omega$, and up to a linear change of coordinates which renders $A(x^0) = \text{const.}\, I$, $u(x) - u(x^0)$ is asymptotic to a homogeneous harmonic polynomial of $x - x^0$, and this asymptotics carries over to first order derivatives. From this basic fact, one can derive that if u is nonidentically constant, then its critical points are isolated. Moreover, if x^0 is a zero of multiplicity m for ∇u, then the level set $\{x \mid u(x) = u(x^0)\}$ is composed, near x^0, by exactly $m+1$ simple arcs intersecting at x^0 only. Next, it is possible to evaluate the number, and the multiplicities, of critical points of a solution in terms of properties of its Dirichlet data [A1], [A2], or of other types of boundary data [AM1]. Such results have also been generalized to weak solutions u to (1.2) when the coefficient matrix A is merely bounded measurable [AM2]. In this case, however, the notion of critical point requires some adjustments, since the gradient of a solution may be discontinuous and, consequently, speaking of zeroes of the gradient may become meaningless. This difficulty is circumvented through the reduction of equation (1.2) to a first order elliptic system of Beltrami type, which enables to represent locally a solution u to (1.2) in the form $u = h \circ \chi$ where h is a harmonic function and χ is a quasiconformal mapping [BJS]. In view of such a representation, one can introduce the notion of geometric critical point saying that x^0 is a geometric critical point of u if $\chi(x^0)$ is a critical point of h. Moreover, one can define the geometric index of u at x^0 as the multiplicity of the critical point $\chi(x^0)$ for h. With such definitions, it is possible to extend to the case of equations with L^∞ coefficients A the evaluations of the number of geometric critical points (see [AM2] for details).

One recent application of such type of results has been the generalization of a theorem of Radó. Radó's theorem says the following. Let B the unit disk, G a bounded convex domain in $\mathbf{R}^2$ and Φ: $\partial B \to \partial G$ a homeomorphism from the unit circle onto the boundary of G. If U: $B \to G$ is the the mapping whose components u_1, u_2 are the harmonic functions having, as Dirichlet data, the components ϕ^1, ϕ^2 of Φ, then U is also a homeomorphism of the unit disk B onto G. Such a theorem was originally stated by Radó [R], subsequently proven by Kneser [K] and also by Choquet [C]. See Duren–Hengartner [DH], Lyzzaik [Ly] and Laugesen [La] for more recent developments of the subject and details on the history of this theorem. See also Schoen–Yau [SY], Jost [J] for generalizations to harmonic mappings between Riemannian manifolds. Here we wish to focus on a different recent generalization of Radó's theorem due to Bauman, Marini and Nesi [BMN] which, in its essence, says as follows. Let B, G and Φ as above, and let $U = (u^1, u^2)$, instead of being harmonic, be such that u^1, u^2 are the solutions to (1.2) in B whose Dirichlet data are φ^1, φ^2, respectively. Also in this case, Bauman, Marini and Nesi are able to prove that U is a homeomorphism of B onto G. The gist of their proof stands in showing that, for every nontrivial combination $u_\theta = u^1 \cos\theta + u^2 \sin\theta$, $0 \le \theta < \pi$, of the components of U, u_θ has no critical points inside of Ω, which in turn implies the non vanishing of the Jacobian determinant $|\partial(u^1, u^2)/\partial(x^1, x^2)|$. Indeed, such absence of critical points for u_θ can be inferred by the above mentioned evaluations of the number of critical points.

Let us now return to the nonlinear equation (1.1). In this case, the study of critical points of solutions has been developed, since now, only for the case when $A = I$, that is for the well-known p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \qquad \text{in } \Omega. \tag{1.3}$$

The study was initiated by Bojarski and Iwaniec [BI], who proved, for the case $p > 2$, that critical points are isolated, by methods of quasiconformal mappings. Other proofs were given for any $1 < p < \infty$, by Alessandrini [A3] where local estimates on the number of critical points were obtained by real analytic methods, and by Manfredi [Ma] who developed the methods by [BI]. See also Lewis [Le] where an estimate on the number of critical points in terms of the Dirichlet data is given.

A remarkable application of the discreteness of critical points that was found by Manfredi is the validity of the strong comparison principle for solutions to (1.3) [Ma, Theorem 2]. Namely if $u^1 \neq u^2$, $u^1 \leq u^2$, are solutions to (1.3) then they satisfy the strict inequality $u^1 < u^2$ everywhere. This result is quite peculiar, because, when the space dimension is bigger than two, such comparison principle is still unknown for the p-Laplacian.

In this paper we prove that critical points of non-constant solutions to (1.1) are isolated and that the local topological structure of the level lines is the same as the one of harmonic functions (Proposition 3.3). Moreover we prove (Theorem 3.1) estimates on the number of critical points in terms of the number of oscillations of the Dirichlet data, which parallel those known for linear equations [A1, Theorem 1.2], [A2, Theorem 1.1], [AM2, Theorem 2.7].

As applications of these results, we prove:

(i) a strong comparison principle (Theorem 4.1) for equation (1.1), which extends the above mentioned result of Manfredi;
(ii) a generalization of Radó's theorem for pairs u^1, u^2 of solutions to (1.1).

Let us stress here the remarkable fact that, when $p \neq 2$, a linear combination $u_\theta = u^1 \cos\theta + u^2 \sin\theta$ need not be a solution to equation (1.1). Nevertheless, we are able to show that, away from the critical points of u^1, u^2, u_θ is a solution to a suitable divergence structure linear elliptic equation, and thus, in the end, we can show that under the above mentioned hypotheses on Φ and G, for every θ the function u_θ has no interior critical points.

In order to obtain the above mentioned results, we have found it convenient to single out the relevant topological properties of level lines which are typical of harmonic functions in two variables and which are known to be shared by solutions to linear elliptic equations. To this purpose, we define the class $\mathscr{S}(\Omega)$ of continuous functions in Ω whose level lines are locally finite union of simple arcs with discrete pairwise intersections, and which satisfy the strong maximum principle (in the sense that they do not admit interior relative extremum points), see Definition 2.1. Since no differentiability is involved in such a definition, it is necessary to introduce a generalized notion of critical point. Such a notion indeed extends (see Corollary 2.6) the one of geometric critical point introduced in [AM2], and for this reason we shall continue to use such a name. We shall prove (Theorem 2.3) that it is possible to estimate the number of the geometric critical points of a function $u \in \mathscr{S}(\Omega)$ in terms of the number of oscillations of its boundary values $g = u|_{\partial\Omega}$, in the same fashion as it was known for solutions to linear elliptic equations. By this device, we shall be able to prove our estimate on the number of critical points of a solution u to (1.1), just by proving that u belongs to the class $\mathscr{S}(\Omega)$. Let us remark that this approach although being different presents some resemblances with the theory of pseudoharmonic functions by Morse (see, for instance, [Mo] and [JM]).

The plan of the paper is as follows. In Section 2 we introduce the class $\mathscr{S}(\Omega)$, and we prove the estimate on the number of geometric critical points of functions $u \in \mathscr{S}(\Omega)$ in Theorem 2.3. Finally we show how $\mathscr{S}(\Omega)$ contains relevant subclasses formed by the components of quasiregular mappings, Proposition 2.5, and by solutions to linear elliptic equations, Corollary 2.6, Proposition 2.7. The main result of Section 3 is Theorem 3.1, which gives the estimate on the number of critical points of solutions to (1.1) in terms of the Dirichlet data. We also prove, in Proposition 3.3, a result on the local behavior of solutions to (1.1) which, in particular, implies that critical points are isolated. In Section 4 we prove (Theorem 4.1) the generalization of Manfredi's strong comparison principle. Finally, in Section 5, we prove Theorem 5.1 which provides the generalization of Radó's theorem for pairs of solutions to (1.1).

2. The class $\mathscr{S}(\Omega)$ and geometric critical points

Definition 2.1. Let $u \in \mathscr{C}(\Omega)$ and let Γ_t, for every $t \in \mathbf{R}$, denote the level line $\{z \in \Omega \mid u(z) = t\}$. We shall say that u belongs to the class $\mathscr{S}(\Omega)$ if it verifies the following conditions:

(S.1) For every $z_0 \in \Omega$, there exist a neighborhood $U \subset \Omega$ of z_0 and an integer $I(z_0)$, $0 \le I(z_0) < \infty$, such that the set $\Gamma_{u(z_0)} \cap U$ is made of $I(z_0) + 1$ simple arcs, whose pairwise intersection consists of $\{z_0\}$ only.

(S.2) u does not have interior maxima nor minima (strong maximum principle).

Moreover we shall say that $z_0 \in \Omega$ is a *geometric critical point of* u if

$$I(z_0) \ge 1.$$

In such a case we shall say that $I(z_0)$ is the *geometric index of* u *at* z_0.

Remark 2.2. Notice that, for any $u \in \mathscr{S}(\Omega)$, property (S.1) implies that the geometric critical points of u are isolated. Moreover, by virtue of property (S.2), u is non-constant on every open subset of Ω. Hence, if Ω is simply connected for every $t \in \mathbf{R}$, the level line Γ_t does not contain closed curves.

Let us also observe that if in addition $u \in \mathscr{C}^1(\Omega)$ and has isolated critical points then the geometric index coincides with the usual notion of index, namely the index $I(z_0, u)$ of a critical point $z_0 \in \Omega$ is defined as the limit

$$I(z_0, u) = \lim_{r \to 0} I\big(B_r(z_0), u\big),$$

where, for each $D \subset\subset \Omega$ such that ∂D is smooth and contains no critical points of u, $I(D, u)$ is given by

$$I(D, u) = -\frac{1}{2\pi} \int_{\partial D} d \arg \nabla u.$$

This, in fact, is a consequence of [AM1, Lemma 3.1] which provides a classification of isolated critical points of $\mathscr{C}^1$ functions in the plane.

Theorem 2.3. *Let Ω be a bounded simply connected domain in $\mathbf{R}^2$ whose boundary $\partial\Omega$ is a simple closed curve. Let $g \in \mathscr{C}(\partial\Omega)$ be such that*

(Σ_N) *$\partial\Omega$ can be split into $2N$ consecutive arcs on which g is alternatively a nonincreasing and nondecreasing function.*

If $u \in \mathscr{C}(\overline{\Omega}) \cap \mathscr{S}(\Omega)$ is such that $u|_{\partial\Omega} = g$ then the geometric critical points of u are finite in number and their geometric indices satisfy

$$\sum_{z \in \Omega} I(z) \leq N - 1.$$

Remark 2.4. Notice that the above assumption (Σ_N) about g is equivalent to saying that the set of points of relative maxima of g on $\partial\Omega$ has at most N connected components. Likewise, relative maxima can be replaced by relative minima.

Proof of Theorem 2.3. The proof we give here is based on ideas appearing in [A2, Theorem 1.1], which are re-elaborated in a simplified version.

The present approach is based on the following observation. Let A be an open subset of Ω such that $\partial A \subset \partial\Omega \cup \Gamma_t$ for some $t \in \mathbf{R}$. We shall call such A a *cell.* We have that, within $\bar{A}$, u attains either a maximum or a minimum strictly larger or, respectively, smaller than t. Therefore, by (S.2), such an extremum is attained at some point in the interior of $\partial A \cap \partial\Omega$. Now, if we consider any family $\mathscr{F}$ of pairwise disjoint cells, by our assumption on g, we have that $\mathscr{F}$ can have at most $2N$ elements.

For any connected component L of a level line Γ_t, we set $s(L) := \sum_{z \in L} I(z)$. We have that $\Omega \setminus L$ is made of $2\big(s(L) + 1\big)$ connected components. This can be proven by the arguments in [A2, Lemmas 1.1, 1.2, 1.3], see also [S, Lemmas 2.8, 2.9]. An alternative proof may be outlined as follows.

The number of connected components of $\Omega \setminus L$ remains unchanged if one identifies to a single point each simple arc in L which joins any two contiguous geometric critical points in L.

Therefore, it suffices to consider the case when L contains only a single geometric critical point (or none). In such a case L is composed of $s(L)+1$ simple arcs all intersecting just once in the single geometric critical point (if it exists) which split Ω into $2\bigl(s(L)+1\bigr)$ connected components.

Since such connected components form a family of disjoint cells we have that $s(L) \le N-1$. In particular L contains at most a finite number of geometric critical points.

Consider now any finite family $\{L_i\}_{1\le i\le n}$ of connected components of level lines Γ_{t_i}, $i=1,\dots,n$. We must show that $\sum_{i=1}^n s(L_i) \le N-1$. It suffices to prove that we can select, among the set of connected components of $\Omega \setminus \bigcup_{i=1}^n L_i$, a family $\mathscr{F}_n$ of at least $2(1+\sum_{i=1}^n s\bigl(L_i)\bigr)$ disjoint cells. We have just proved that this is the case when $n=1$ and we complete the proof by induction on n. Take $n>1$ and choose a component, say L_n, such that $L_1,\dots,L_{n-1}$ all lie in the same connected component of $\Omega\setminus L_n$ (the existence of such L_n follows from a straightforward induction argument). Let A be the connected component of $\Omega\setminus\bigcup_{i=1}^{n-1} L_i$ that contains L_n. Denote by $\mathscr{G}$ the family of connected components of $\Omega\setminus L_n$ contained in A and define $\mathscr{F}_n$ as follows

$$\mathscr{F}_n := \mathscr{G} \cup (\mathscr{F}_{n-1}\setminus\{A\}).$$

Since

$$\#\mathscr{G} = 1+2s(L_n), \qquad \#(\mathscr{F}_{n-1}\setminus\{A\}) \ge \#\mathscr{F}_{n-1}-1 \ge 1+2\sum_{i=1}^{n-1} s(L_i)$$

we have

$$\#\mathscr{F}_n \ge 2\biggl(1+\sum_{i=1}^{n} s(L_i)\biggr)$$

and all elements of $\mathscr{F}_n$ are pairwise disjoint cells. □

Let us now highlight an important subclass of $\mathscr{S}(\Omega)$. To do this it is useful to identify $\mathbf{R}^2$ and $\mathbf{C}$ in the standard way.

Proposition 2.5. *Let f be a non-constant locally quasiregular mapping on Ω. Then $\operatorname{Re} f$ and $\operatorname{Im} f$, the real and the imaginary parts of f, belong to $\mathscr{S}(\Omega)$.*

Proof. This result is a rather straightforward consequence of the well-known Ahlfors–Bers factorization theorem [AB]. For the sake of completeness we outline a proof. Since (S.1) and (S.2) are local conditions, possibly replacing Ω by a compact subset we are allowed to suppose f quasiregular in Ω. The theory of quasiregular mappings in dimension two (see also [BJS, Chapter II.6]) ensures the representation by factorization $f = F\circ\chi$ where $\chi\colon \Omega\to B_1(0)$ is quasiconformal and $F\colon B_1(0)\to\mathbf{C}$ is holomorphic. Since χ is a homeomorphism, it is enough to observe that being $\operatorname{Re} F$ and $\operatorname{Im} F$ harmonic they belong to $\mathscr{S}(B_1(0))$. This is easily seen, noticing that, for each $z_0 \in \Omega$, $F(z) - F(z_0)$ is asymptotic as $z \to z_0$ to a homogeneous polynomial $c(z - z_0)^n = |c|\rho^n(\cos(n\theta + \theta_0) + i\sin(n\theta + \theta_0))$ where $c \in \mathbf{C} \setminus \{0\}$, $n \geq 1$, ρ, θ are the polar coordinates centered at z_0, and θ_0 is given by $c = |c|e^{i\theta_0}$. □

From Proposition 2.5 and Theorem 2.1 in [AM2], where it is proved that a solution to a linear uniformly elliptic equation in divergence form is the real part of a quasiregular mapping, we get

Corollary 2.6. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a non-constant solution to*

$$\operatorname{div}(A\nabla u) = 0 \qquad \text{in } \Omega,$$

where the symmetric matrix $A \in L^\infty_{\mathrm{loc}}(\Omega)$ is uniformly elliptic on the compact subsets of Ω. Then u belongs to $\mathscr{S}(\Omega)$.

The following proposition provides a variant to Corollary 2.6, in that it deals with nondivergence elliptic equations. Although we shall not make use of it in the sequel, it may be interesting to note that it could provide an alternative approach to the proof of Proposition 3.3 below. We insert it here, for the sake of completeness.

Proposition 2.7. *Let $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ be a non-constant solution to*

$$a_{ij}u_{x_ix_j} + b_iu_{x_i} = 0 \qquad \text{almost everywhere in } \Omega,$$

where the matrix $A = (a_{ij}) \in L^\infty_{\mathrm{loc}}(\Omega)$ is uniformly elliptic on compact sets of Ω and $b_1, b_2 \in L^\infty_{\mathrm{loc}}(\Omega)$. Then u belongs to $\mathscr{S}(\Omega)$.

Proof. By the results of Bers and Nirenberg (see [BN] and also [BJS]) we have that $u \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some α, $0 < \alpha < 1$. Moreover, its critical points are isolated and have positive index. Therefore by [AM1, Lemma 3.1] u satisfies (S.1). Moreover (S.2) follows from Hopf's lemma (see for instance [GT, Theorem 9.6]). □

3. Critical points for the anisotropic p-Laplacian

From now on we shall assume that the coefficient matrix $A = (a_{ij})$ is symmetric and satisfies the following conditions. For given constants λ, L, $0 < \lambda \leq 1$, $L > 0$ we assume

(3.1) $$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda^{-1}|\xi|^2 \text{ for every } \xi \in \mathbf{R}^2 \text{ and } x \in \mathbf{R}^2,$$

(3.2) $$|A(x) - A(y)| \leq L|x - y| \text{ for every } x, y \in \mathbf{R}^2.$$

Theorem 3.1. *Let Ω be a bounded simply connected domain in $\mathbf{R}^2$ verifying an exterior cone condition and let $\partial\Omega$ be a simple closed curve. Let A satisfy (3.1) and (3.2). Let $g \in \mathscr{C}(\partial\Omega)$ be a non-constant function satisfying (Σ_N). If $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ is the solution to (1.1) satisfying the Dirichlet condition $u|_{\partial\Omega} = g$, then the number of geometric critical points of u in Ω, when counted according to their geometric index, is less than or equal to $N - 1$.*

Remark 3.2. The exterior cone condition on Ω stated above has the only motivation of providing a sufficient condition, both geometrically elementary and independent of p, for the existence (and uniqueness) of a solution to the Dirichlet problem which is continuous at every boundary point. More refined conditions based on a Wiener test could be introduced and for this matter we refer to [MZ] and in particular Corollary 6.22.

Before proceeding to the proof of Theorem 3.1 let us perform some formal calculations. Computing the divergence in (1.1), we get

$$\begin{aligned}\operatorname{div}(&|A\nabla u\cdot\nabla u|^{(p-2)/2}A\nabla u)\\ &=|A\nabla u\cdot\nabla u|^{(p-2)/2}\operatorname{div}(A\nabla u)+\nabla(|A\nabla u\cdot\nabla u|^{(p-2)/2})\cdot A\nabla u\\ &=|A\nabla u\cdot\nabla u|^{(p-2)/2}\left[\operatorname{div}(A\nabla u)+\frac{p-2}{2}\frac{\nabla(A\nabla u\cdot\nabla u)\cdot A\nabla u}{A\nabla u\cdot\nabla u}\right].\end{aligned}$$

Therefore, formally speaking, u satisfies the nondivergence uniformly elliptic equation (as we shall see later on in the course of the proof of Proposition 3.3)

$$b_{ij}u_{x_ix_j}+c_iu_{x_i}=0,$$

where

$$b_{ij}=a_{ij}+(p-2)\frac{(A\nabla u)_i(A\nabla u)_j}{A\nabla u\cdot\nabla u}, \tag{3.3}$$

and

$$c_i=a_{ji,x_j}+\frac{p-2}{2}\frac{a_{ij}A_{x_j}\nabla u\cdot\nabla u}{A\nabla u\cdot\nabla u}.$$

Thus it is expected that u has indeed $\mathscr{C}^{1,\alpha}_{\text{loc}}$ regularity and hence, for u, the index and geometric index should coincide. Such formal considerations are justified by a regularization argument, whose details are provided by the following proposition. Let us remark that the local $\mathscr{C}^{1,\alpha}$ regularity of a solution to (1.1) holds in any space dimension $n\geq 2$ ([To], [DB]).

Proposition 3.3. *Let A satisfy the hypotheses in Theorem 3.1 and let $u\in W^{1,p}_{\text{loc}}(\Omega)$ be a solution to (1.1). There exist constants α,k, $0<\alpha\leq 1$, $0\leq k<1$ only depending on p, λ and L such that for every $G\subset\subset\Omega$ there exist $s,h\in\mathscr{C}^{\alpha}(\overline{G},\mathbf{C})$ such that the following representation holds in G:*

$$u_z=e^sh, \tag{3.4}$$

and h is a k-quasiregular mapping in G.

Notice that here, and in the sequel, we denote $\partial_z=\frac{1}{2}(\partial_{x_1}-i\partial_{x_2})$, $\partial_{\bar z}=\frac{1}{2}(\partial_{x_1}+i\partial_{x_2})$.

Remark 3.4. We observe that the representation (3.4) automatically implies the $\mathscr{C}^{1,\alpha}_{\rm loc}$ regularity of u and also that, if we assume Ω connected and u non-constant, then its critical points are isolated and have positive index.

Proof of Proposition 3.3. Let D be such that $G \subset\subset D \subset\subset \Omega$ and ∂D is $\mathscr{C}^\infty$-smooth. By standard mollification let us construct functions $g^\varepsilon \in \mathscr{C}^\infty(\overline{D})$, $\varepsilon > 0$, such that g^ε converges to u in $W^{1,p}(D)$ as $\varepsilon \to 0$, and symmetric matrices A^ε with $\mathscr{C}^\infty$ entries such that A^ε satisfies conditions (3.1), (3.2) and A^ε converges to A in $W^{1,q}(D)$ as $\varepsilon \to 0$, for every $q < \infty$. For every $\varepsilon > 0$, we consider $u^\varepsilon \in W^{1,p}(D)$ as the weak solution to

$$\begin{cases} \operatorname{div}((|A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon| + \varepsilon)^{(p-2)/2} A^\varepsilon \nabla u^\varepsilon) = 0 & \text{in } D, \\ u^\varepsilon - g^\varepsilon \in W_0^{1,p}(D). \end{cases} \tag{3.5}$$

For the regularized boundary value problem (3.5), standard elliptic regularity theory applies (see for instance [GT, Theorems 6.19, 11.5]) and it ensures $u^\varepsilon \in \mathscr{C}^\infty(\overline{D})$ for every $\varepsilon > 0$.

A computation for u^ε analogous to the one performed for u, yields

$$\operatorname{div}\big((|A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon| + \varepsilon)^{(p-2)/2} A^\varepsilon \nabla u^\varepsilon\big) = (|A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon| + \varepsilon)^{(p-2)/2} [b^\varepsilon_{ij} u^\varepsilon_{x_i x_j} + c^\varepsilon_i u^\varepsilon_{x_i}],$$

where

$$b^\varepsilon_{ij} = a^\varepsilon_{ij} + (p-2) \frac{(A^\varepsilon \nabla u^\varepsilon)_i (A^\varepsilon \nabla u^\varepsilon)_j}{|A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon| + \varepsilon}, \qquad i, j = 1, 2,$$

and

$$c^\varepsilon = (c^\varepsilon_i) = \left(a^\varepsilon_{ji,x_j} + \frac{p-2}{2} \frac{a^\varepsilon_{ij} A^\varepsilon_{x_j} \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon| + \varepsilon} \right).$$

This allows us to rewrite (3.5) in the form

$$b^\varepsilon_{ij} u^\varepsilon_{x_i x_j} + c^\varepsilon_i u^\varepsilon_{x_i} = 0.$$

Let us study the ellipticity of

$$B^\varepsilon = (b^\varepsilon_{ij})^2_{i,j=1} = A^\varepsilon + (p-2) \frac{(A^\varepsilon \nabla u^\varepsilon) \otimes (A^\varepsilon \nabla u^\varepsilon)}{|A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon| + \varepsilon},$$

where $v \otimes w$ denotes, for $v, w \in \mathbf{R}^2$, the matrix $(v_i w_j)^2_{i,j=1}$. Since for each symmetric matrix $\mathscr{M}$, the equalities $\mathscr{M}(v \otimes w) = (\mathscr{M} v) \otimes w$ and $(v \otimes w)\mathscr{M} = v \otimes (\mathscr{M} w)$ hold, we have

$$B^\varepsilon = \sqrt{A^\varepsilon} \left(I + (p-2) \frac{(\sqrt{A^\varepsilon}\, \nabla u^\varepsilon) \otimes (\sqrt{A^\varepsilon}\, \nabla u^\varepsilon)}{|\sqrt{A^\varepsilon}\, \nabla u^\varepsilon|^2 + \varepsilon} \right) \sqrt{A^\varepsilon}.$$

Observing that

$$0 \le \frac{(\sqrt{A^\varepsilon}\,\nabla u^\varepsilon) \otimes (\sqrt{A^\varepsilon}\,\nabla u^\varepsilon)}{|\sqrt{A^\varepsilon}\,\nabla u^\varepsilon|^2 + \varepsilon}\xi \cdot \xi \le |\xi|^2 \qquad \text{for every } \xi \in \mathbf{R}^2,$$

we get

$$\begin{aligned}\min\{1, p-1\}|\xi|^2 &\le \left(I + (p-2)\frac{(\sqrt{A^\varepsilon}\,\nabla u^\varepsilon) \otimes (\sqrt{A^\varepsilon}\,\nabla u^\varepsilon)}{|\sqrt{A^\varepsilon}\,\nabla u^\varepsilon|^2 + \varepsilon}\right)\xi \cdot \xi \\ &\le \max\{1, p-1\}|\xi|^2.\end{aligned}$$

This shows that B^ε satisfies an ellipticity condition like (3.1) when λ is replaced by a constant $0 < \tilde{\lambda} \le 1$ only depending on p and λ. Let us stress that $\tilde{\lambda}$ is independent of ε. Similarly we obtain that c^ε satisfies a uniform L^∞ bound independent of ε. By well-known computations (see for instance [BJS]) we have, setting $f^\varepsilon := u^\varepsilon_z$,

$$f^\varepsilon_{\bar{z}} = \mu^\varepsilon f^\varepsilon_z + \nu^\varepsilon \overline{f^\varepsilon_z} + \gamma^\varepsilon f^\varepsilon + \delta^\varepsilon \overline{f^\varepsilon}$$

with $\mu^\varepsilon, \nu^\varepsilon, \gamma^\varepsilon, \delta^\varepsilon \in L^\infty(D)$ such that

$$|\mu^\varepsilon| + |\nu^\varepsilon| \le k < 1, \qquad |\gamma^\varepsilon| + |\delta^\varepsilon| \le K < \infty \qquad \text{in } D,$$

where k and K only depend on p, λ and L.

Thus, for each $\varepsilon > 0$, the Bers–Nirenberg factorization (see [BN] or [BJS, II.6]) applies to f^ε. That is, $f^\varepsilon = e^{s^\varepsilon} h^\varepsilon$, where h^ε is k-quasiregular in D and s^ε is α-Hölder in $\overline{D}$ for some α, $0 < \alpha < 1$, independent of ε, $\|s^\varepsilon\|_{\mathscr{C}^{0,\alpha}(D)} \le C = C(k, K)$.

The ellipticity of B^ε also implies (see for instance Talenti [Ta]) a $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}$ estimate for each u^ε, independent of ε. More precisely

$$\|\nabla u^\varepsilon\|_{\infty;G} + |\nabla u^\varepsilon|_{\alpha;G} \le C\|\nabla u^\varepsilon\|_{p,D} \tag{3.6}$$

with α and C independent of ε.

By the uniqueness of the $W^{1,p}$ solution to the Dirichlet problem for equation (1.1) we have that there exists a sequence $\varepsilon_n \to 0$ such that u^{ε_n} converges to u weakly in $W^{1,p}(D)$ and, by (3.6) and the Ascoli–Arzelà theorem, possibly choosing a smaller α, $0 < \alpha < 1$, is also convergent in $\mathscr{C}^{1,\alpha}(G)$. Possibly passing to a subsequence, also s^{ε_n} converges to $s \in \mathscr{C}^\alpha(G, \mathbf{C})$ and $h^{\varepsilon_n} = e^{-s^{\varepsilon_n}} f^{\varepsilon_n} \to h$ in $\mathscr{C}^\alpha(G, \mathbf{C})$. Thus, h being the uniform limit of a sequence of k-quasiregular mappings, we also have that h is k-quasiregular [LV, I.4.9] and $u_z = e^s h$ in G. □

Proof of Theorem 3.1. Let $u^\varepsilon \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}(\overline{\Omega})$, $\varepsilon > 0$, be the solution to the regularized equation

$$\operatorname{div}((|A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon| + \varepsilon)^{(p-2)/2} A^\varepsilon \nabla u^\varepsilon) = 0 \qquad \text{in } \Omega$$

with Dirichlet data

$$u^\varepsilon = g \qquad \text{on } \partial\Omega,$$

where A^ε is as in Proposition 3.3. In view of the theory in [MZ] we have that u^ε converges to u in $\mathscr{C}(\overline{\Omega})$, and also in the $\mathscr{C}^{1,\alpha}_{\text{loc}}(\Omega)$ sense. Let us fix an open subset $D \subset\subset \Omega$ with $\mathscr{C}^\infty$-boundary such that u does not have critical points on ∂D. Since the critical points of u are isolated, it suffices to prove, by the arbitrariness of D, that the number of critical points of u in D, counted according to their index, is $\leq N-1$.

Since u^{ε_n} is smooth, its index and geometric index coincide. Moreover, by construction, we can consider u^{ε_n} as the solution of the elliptic equation in divergence form

$$\operatorname{div}(\tilde{A}_n \nabla u^\varepsilon) = 0 \qquad \text{in } \Omega,$$

where

$$\tilde{A}_n = (|A^{\varepsilon_n}\nabla u^{\varepsilon_n} \cdot \nabla u^{\varepsilon_n}| + \varepsilon)^{(p-2)/2} A^{\varepsilon_n},$$

for which the hypotheses of Corollary 2.6 apply. Therefore

$$I(D, u^{\varepsilon_n}) \leq N-1.$$

By the $\mathscr{C}^1(\overline{D})$ convergence of u^{ε_n} to u and since $|\nabla u|$ is uniformly bounded away from 0 on ∂D, we have that $\arg\nabla u^{\varepsilon_n}$ converges uniformly to $\arg\nabla u$ as $n\to\infty$ and that u^{ε_n}, for n large enough, does not have critical points on ∂D. Hence we obtain

$$I(D,u) = \int_{\partial D} d\arg\nabla u = \lim_{n\to\infty}\int_{\partial D} d\arg\nabla u^{\varepsilon_n} = I(D,u^{\varepsilon_n}) \leq N-1. \ \square$$

Remark 3.5. It is known (see [HKM, Theorem 6.5]) that a non-constant solution u to (1.1) on a simply connected domain verifies a maximum principle in the (S.2)-form. Since, in Proposition 3.3, we have proved that $u \in \mathscr{C}^{1,\alpha}_{\text{loc}}(\Omega)$ and its critical points are isolated, in view of [AM1, Lemma 3.1], we also obtain that u belongs to the class $\mathscr{S}(\Omega)$.

4. The strong comparison principle

Theorem 4.1 (Strong comparison principle). *Let Ω be a bounded domain in $\mathbf{R}^2$, let A satisfy* (3.1), *and* (3.2), *and let $u_0, u_1 \in W^{1,p}_{\text{loc}}(\Omega)$ be two distinct solutions to* (1.1). *If $u_0 \leq u_1$ in Ω, we have $u_0 < u_1$ everywhere in Ω.*

We first prove the following lemma.

Lemma 4.2. *Let $F\colon \mathbf{R}^4 \setminus \{0\} \to \mathbf{R}$ be defined by*

$$F(X_0, X_1) = \int_0^1 |tX_0 + (1-t)X_1|^{p-2}\, dt$$

for each pair (X_0, X_1) of vectors in $\mathbf{R}^2$. Then there exist two constants $0 < c_p \leq C_p < +\infty$ only depending on p such that

$$c_p(|X_0|^2 + |X_1|^2)^{(p-2)/2} \leq F(X_0, X_1) \leq C_p(|X_0|^2 + |X_1|^2)^{(p-2)/2} \tag{4.1}$$

for every $(X_0, X_1) \in \mathbf{R}^4 \setminus \{0\}$.

Proof. Let G_1 denote the subset of $\mathbf{R}^4 \setminus \{0\}$ consisting of pairs (X_0, X_1) such that 0 does not belong to the segment $[X_0, X_1]$. If $(X_0, X_1) \in G_1$ then

$$0 < \operatorname{dist}(0, [X_0, X_1]) \leq |tX_0 + (1-t)X_1| \leq \max\{|X_0|, |X_1|\} < +\infty.$$

Notice that $\operatorname{dist}(0, [X_0, X_1])$ and $\max\{|X_0|, |X_1|\}$ are continuous functions of (X_0, X_1) in G_1. We get,

$$0 < \big(\operatorname{dist}(0, [X_0, X_1])\big)^{p-2} \leq F(X_0, X_1) \leq \big(\max\{|X_0|, |X_1|\}\big)^{p-2} < \infty \text{ when } p \geq 2,$$

and

$$0 < \big(\max\{|X_0|, |X_1|\}\big)^{p-2} \leq F(X_0, X_1) \leq \big(\operatorname{dist}(0, [X_0, X_1])\big)^{p-2} < \infty \text{ when } p \leq 2.$$

Consequently, there exist two continuous and positive functions ϕ_1 and Φ_1 on G_1, only depending on p, such that $\phi_1 \leq F \leq \Phi_1$ on G_1.

Let G_2 denote the subset of $\mathbf{R}^4 \setminus \{0\}$ consisting of the pairs (X_0, X_1) such that $X_0 \neq X_1$. Notice that on G_2 we have a uniquely determined, continuous function τ defined, for each pair (X_0, X_1), by

$$\big|\tau(X_0, X_1)X_0 + \big(1 - \tau(X_0, X_1)\big)X_1\big| = \min_{t \in \mathbf{R}} |tX_0 + (1-t)X_1|.$$

Fix $(X_0, X_1) \in G_2$ and set $\tau = \tau(X_0, X_1)$. Since $X_0 - X_1$ is perpendicular to $\tau X_0 + (1-\tau)X_1$, then, for all $t \in [0,1]$,

$$\begin{aligned} |t-\tau|\,|X_0 - X_1| &\leq |(t-\tau)(X_0 - X_1) + \tau X_0 + (1-\tau)X_1| \\ &= |tX_0 + (1-t)X_1| \leq \max\{|X_0|, |X_1|\}. \end{aligned}$$

Let us remark that

$$\int_0^1 |t-\tau|^{p-2}\,dt = \int_{-\tau}^{1-\tau} |s|^{p-2}\,ds;$$

therefore, being $p - 2 > -1$, $\int_0^1 |t-\tau|^{p-2}\,dt$ is a continuous function of τ and, consequently, of (X_0, X_1) in G_2. As we did for G_1, also on G_2 we can bound F from above and below by two continuous positive functions, say ϕ_2, Φ_2, only depending on p.

Since F is homogeneous of degree $p-2$, we have

$$F(X_0, X_1) = (|X_0|^2 + |X_1|^2)^{(p-2)/2} F(\xi_0, \xi_1),$$

where

$$\xi_i = \frac{X_i}{(|X_0|^2 + |X_1|^2)^{1/2}}, \qquad i = 0, 1,$$

in such a way that $(\xi_0, \xi_1) \in S^3$, where S^3 denotes the unit sphere in $\mathbf{R}^4$.

Let $\{\alpha_1, \alpha_2\}$ be a partition of unity on S^3 subordinate to the covering $\{G_1, G_2\}$. Then

$$\alpha_1\phi_1 + \alpha_2\phi_2 \le F \le \alpha_1\Phi_1 + \alpha_2\Phi_2 \qquad \text{on } S^3,$$

and the desired estimate (4.1) holds with

$$c_p = \min_{S^3}(\alpha_1\phi_1 + \alpha_2\phi_2), \qquad C_p = \max_{S^3}(\alpha_1\Phi_1 + \alpha_2\Phi_2). \ \square$$

Proof of Theorem 4.1. Without loss of generality we can assume that either u_0 or u_1 is non-constant. Set $v = u_1 - u_0$ and, for $t \in [0,1]$, $u_t = tu_1 + (1-t)u_0$.

For each test function $\varphi \in W_c^{1,p}(\Omega)$ we have

$$\begin{aligned} 0 &= \int_\Omega \big(|A\nabla u_1 \cdot \nabla u_1|^{(p-2)/2} A\nabla u_1 - |A\nabla u_0 \cdot \nabla u_0|^{(p-2)/2} A\nabla u_0\big) \cdot \nabla\varphi \\ &= \int_\Omega \left(\int_0^1 \frac{d}{dt}\big(|A\nabla u_t \cdot \nabla u_t|^{(p-2)/2} A\nabla u_t\big) \cdot \nabla\varphi \right) dt. \end{aligned}$$

A calculation gives us

$$\begin{aligned} &\frac{d}{dt}\big(|A\nabla u_t \cdot \nabla u_t|^{(p-2)/2} A\nabla u_t\big) \\ &= |A\nabla u_t \cdot \nabla u_t|^{(p-2)/2} A\nabla v + (p-2)|A\nabla u_t \cdot \nabla u_t|^{(p-4)/2}(A\nabla u_t \cdot \nabla v) A\nabla u_t \\ &= |A\nabla u_t \cdot \nabla u_t|^{(p-2)/2}(A + (p-2)\frac{A\nabla u_t \otimes A\nabla u_t}{A\nabla u_t \cdot \nabla u_t})\nabla v, \end{aligned}$$

and we obtain

$$\int_\Omega \mathscr{A}\nabla v \cdot \nabla\varphi = 0 \qquad \text{for all } \varphi \in W_c^{1,p}(\Omega),$$

where

$$\mathscr{A} = \int_0^1 |A\nabla u_t \cdot \nabla u_t|^{(p-2)/2} \left(A + (p-2)\frac{A\nabla u_t \otimes A\nabla u_t}{A\nabla u_t \cdot \nabla u_t} \right) dt. \tag{4.2}$$

This means that $v \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution to

$$\operatorname{div}(\mathscr{A}\nabla v) = 0 \qquad \text{in } \Omega. \tag{4.3}$$

Notice that

$$\mathscr{A} = \int_0^1 |A\nabla u_t \cdot \nabla u_t|^{(p-2)/2} B_t \, dt,$$

where B_t is the matrix B in (3.3) when u is replaced by u_t. By the same arguments used in the proof of Proposition 3.3 we have

$$\tilde\lambda|\xi|^2 \le B_t\xi \cdot \xi \le \tilde\lambda^{-1}|\xi|^2 \qquad \text{for every } \xi \in \mathbf{R}^2 \text{ and } x \in \mathbf{R}^2,$$

where $\tilde{\lambda}$ only depends on p and on λ. Denoting by λ_1, λ_2 the eigenvalues of $\mathscr{A}$, $\lambda_1 \leq \lambda_2$, we obtain

$$\tilde{\lambda} \int_0^1 |A\nabla u_t \cdot \nabla u_t|^{(p-2)/2} \leq \lambda_1 \leq \lambda_2 \leq \frac{1}{\tilde{\lambda}} \int_0^1 |A\nabla u_t \cdot \nabla u_t|^{(p-2)/2}.$$

By the ellipticity assumption on A and the definition of F given in Lemma 4.2, we obtain

$$\tilde{\lambda}\lambda^{|p-2|} F(\nabla u_0, \nabla u_1) \leq \lambda_1 \leq \lambda_2 \leq \frac{1}{\tilde{\lambda}\lambda^{|p-2|}} F(\nabla u_0, \nabla u_1).$$

Denote by S the set of critical points common to u_0 and u_1. Proposition 3.3 implies that S is discrete. From Lemma 4.2 it follows that the equation (4.3) is uniformly elliptic in each $D \subset\subset \Omega \setminus S$. If such a D is also connected, then we have that v is either strictly positive or identically equal to zero in D. From the arbitrariness of D we obtain that either $v \equiv 0$ or $v > 0$ in $\Omega \setminus S$. In the first case we would have, by continuity, that $v \equiv 0$ in Ω, contrary to the hypothesis that $u_0 \neq u_1$. So the second case is true.

Pick a neighborhood V of $z_0 \in S$ such that $\partial V \cap S = \emptyset$ and let $m > 0$ be the minimum of v on ∂V. The weak maximum principle for the solutions to the equation (1.1) (see for instance [HKM, Theorem 7.6]) leads to $u_0 + m \leq u_1$ in V and, in particular, in z_0.

Then $u_0 < u_1$ everywhere in Ω. □

5. The generalization of Radó's theorem

Theorem 5.1. *Let Ω be an open bounded simply connected domain in $\mathbf{R}^2$ satisfying an exterior cone condition and let $\partial\Omega$ be a simple closed curve, G a bounded convex domain in $\mathbf{R}^2$ and $\Phi = (\phi^1, \phi^2)\colon \partial\Omega \to \partial G$ a sense-preserving homeomorphism. Set $U = (u^1, u^2)$ where, for $i = 1, 2$, $u^i \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ is the solution to*

$$\begin{cases} \operatorname{div}(|A\nabla u^i \cdot \nabla u^i|^{(p-2)/2} A\nabla u^i) = 0 & \text{in } \Omega, \\ u^i|_{\partial\Omega} = \phi^i, \end{cases}$$

and the matrix A satisfies (3.1) *and* (3.2). *Then*

(a) *U is a diffeomorphism from Ω onto G and $\det DU > 0$ in Ω.*

(b) *If, moreover, $\partial\Omega \in \mathscr{C}^{1,\alpha}$, $0 < \alpha < 1$, satisfies an interior sphere condition and $\Phi \in \mathscr{C}^{1,\alpha}(\partial\Omega)$, then U is a diffeomorphism from $\overline{\Omega}$ onto $\overline{G}$ and $\det DU > 0$ in $\overline{\Omega}$.*

Before proving the theorem, let us recall the notion of Brower degree. If $\mathbf{\Psi} \in \mathscr{C}^1(\Omega; \mathbf{R}^2) \cap \mathscr{C}(\overline{\Omega}; \mathbf{R}^2)$ and $y \in \mathbf{R}^2 \setminus \mathbf{\Psi}(\partial\Omega)$ then

$$\deg(\mathbf{\Psi}, \Omega, y) := \int_\Omega f\big(\mathbf{\Psi}(x)\big) \det D\mathbf{\Psi}(x)\, dx$$

for any $f \in \mathscr{C}^\infty(\mathbf{R}^2;\mathbf{R})$ with compact support in the connected component of $\mathbf{R}^2 \setminus \mathbf{\Psi}(\partial\Omega)$ containing y and such that

$$\int_{\mathbf{R}^2} f(x)\,dx = 1.$$

It is known that $\deg(\mathbf{\Psi},\Omega,\cdot\,)$ is integer-valued, constant on connected components of $\mathbf{R}^2 \setminus \mathbf{\Psi}(\partial\Omega)$ and within such domains it only depends on $\mathbf{\Psi}|_{\partial\Omega}$. Moreover, if $\det D\mathbf{\Psi}(x) \neq 0$ for each $x \in \mathbf{\Psi}^{-1}(y)$, we have

$$\deg(\mathbf{\Psi},\Omega,y) = \sum_{x\in\mathbf{\Psi}^{-1}(y)} \operatorname{sgn}\big(\det D\mathbf{\Psi}(x)\big). \tag{5.1}$$

Proof of Theorem 5.1. (a) The first, topological, step shall consist in verifying that $G \subseteq U(\Omega)$. Since Φ is orientation preserving and since, as seen, $\deg(U,\Omega,y)$ only depends on $U|_{\partial\Omega} = \Phi$, we have, for each $y \in G$,

$$\deg(U,\Omega,y) = 1. \tag{5.2}$$

Let $E = U(\Omega)\cap G$. If E is not dense in G, we can pick $f \in \mathscr{C}_0^\infty(G\setminus \overline{E})$ such that $\int_{G\setminus\overline{E}} f(x)\,dx = 1$. Hence we have

$$\int_\Omega \underbrace{f\big(U(x)\big)}_{\equiv 0} \det DU(x)\,dx = 0,$$

which contradicts (5.2). From the continuity of U we deduce, as desired, $G \subseteq U(\Omega)$.

Let us denote, for every $\theta \in [0,\pi[$,

$$u_\theta = u^1\cos\theta + u^2\sin\theta.$$

Notice that the convexity of G implies, for each θ, that the number of connected components of points of maximum of $u_\theta|_{\partial\Omega} = U|_{\partial\Omega}\cdot(\cos\theta,\sin\theta)$ is exactly equal to 1. By Theorem 3.1 we then obtain that $u_0 = u^1$ and $u_{\pi/2} = u^2$ do not have critical points in Ω.

Applying to u_θ the interpolation argument used for v in the proof of Theorem 4.1, we get, in analogy to (4.3) and (4.2), that u_θ is a weak solution to

$$\operatorname{div}(\mathscr{A}_\theta\nabla u_\theta) = 0 \qquad \text{in } \Omega,$$

where

$$\mathscr{A}_\theta = \int_0^1 |A\nabla u_{\theta,t}\cdot\nabla u_{\theta,t}|^{(p-2)/2}\bigg(A + (p-2)\frac{A\nabla u_{\theta,t}\otimes A\nabla u_{\theta,t}}{A\nabla u_{\theta,t}\cdot\nabla u_{\theta,t}}\bigg)\,dt,$$

and $u_{\theta,t} = tu^2 \sin\theta + (1-t)u^1 \cos\theta$ for $t \in [0,1]$.

Since u^1 and u^2 do not have critical points in Ω, the matrix $\mathscr{A}_\theta$, by virtue of Lemma 4.2, is uniformly elliptic on every compact subset of Ω.

By Corollary 2.3 and Remark 3.2 we have that u_θ belongs to $\mathscr{S}(\Omega)$ for each $\theta \in [0,\pi[$. In particular (S.2) implies $U(\Omega) \subseteq G$ since a convex subset of the plane coincides with the intersection of the half-planes containing it. Thus, $U(\Omega) = G$.

Notice that u_θ has a unique component of maximum on $\partial\Omega$. By Theorem 2.3 we then have that $\nabla u_\theta \neq 0$ in Ω for each $\theta \in [0,\pi[$, that is, $\det DU \neq 0$ everywhere in Ω.

By continuity, $\det DU > 0$ in Ω and thus, by (5.1),

$$\#\{U^{-1}(y)\} = 1 \qquad \text{for all } y \in G.$$

(b) Let us introduce, for $i = 1,2$, the regularized solutions $u^{i,\varepsilon}$ associated to u^i as we did in Theorem 3.1. That is, for every $\varepsilon > 0$, $u^{i,\varepsilon}$ is the solution to the regularized equation

$$\operatorname{div}\big((|A^\varepsilon \nabla u^{i,\varepsilon} \cdot \nabla u^{i,\varepsilon}| + \varepsilon)^{(p-2)/2} A^\varepsilon \nabla u^{i,\varepsilon}\big) = 0 \qquad \text{in } \Omega$$

coupled with the Dirichlet condition $u^{i,\varepsilon}|_{\partial\Omega} = \phi^i$.

A regularity result due to Lieberman [Li, Theorem 1] implies that $u^{i,\varepsilon}$ satisfy a uniform $\mathscr{C}^{1,\alpha}(\overline{\Omega};\mathbf{R})$ bound for some α, $0 < \alpha \leq 1$, independent of ε and the same bound applies to u^i, $i = 1,2$.

Next we prove that $\det DU > 0$ on $\partial\Omega$. Let us fix $\theta \in [0,\pi[$ and $x_0 \in \partial\Omega$. If x_0 is a critical point of u_θ, then the derivative of u_θ with respect to the tangent unit vector is zero. Thus x_0 lies in a component of points of maximum or of minimum of u_θ. Otherwise ∂G should have a point of inflection contrary to the hypothesis that G is convex.

Now, let x_0 be a point of, say, minimum for u_θ and let us prove that x_0 is not a critical point for u_θ. Suppose for the moment $\theta = 0$, that is $u_\theta = u^1$. Since $u^{1,\varepsilon}|_{\partial\Omega} = u^1|_{\partial\Omega}$, x_0 is a point of minimum for each $u^{1,\varepsilon}$. Then the Hopf maximum principle (see for instance [GT, Lemma 3.4]) applies to $u^{1,\varepsilon}$. Thus, denoting by ν the inward unit normal to $\partial\Omega$ at x_0, we have

$$\frac{\partial u^{1,\varepsilon}}{\partial \nu}(x_0) \geq C > 0 \qquad \text{for every } \varepsilon,\ 0 < \varepsilon \leq 1,$$

where C is independent of ε.

Thus passing to a limit on a subsequence $\varepsilon_n \searrow 0$ we obtain

$$\frac{\partial u^1}{\partial \nu}(x_0) > 0$$

and thus $\nabla u^1(x_0) \neq 0$. Analogous considerations hold when x_0 is a point of relative maximum and also for u^2. Therefore we have that both ∇u^1 and ∇u^2 never vanish in $\overline{\Omega}$.

Hence the matrix $\mathscr{A}_\theta$ is uniformly elliptic in all of Ω and its entries are in $\mathscr{C}^\beta(\overline{\Omega})$, for each θ in $[0,\pi[$. We are then justified in applying to u_θ, at a point of relative maximum or minimum x_0, the generalized Hopf maximum principle due to Finn and Gilbarg [FG, Lemma 7] which applies to the case of elliptic equations in divergence form with Hölder continuous coefficients. Hence we obtain that the derivative of u_θ along the inward normal to $\partial\Omega$ at x_0 does not vanish. Thus x_0 is not a critical point for u_θ for any $\theta \in [0,\pi[$.

As we did in (a), by the arbitrariness of θ, we deduce that $\det DU \neq 0$ on $\partial\Omega$ and, by continuity, $\det DU > 0$ everywhere in $\overline{\Omega}$. □

References

[AB] Ahlfors, L., and L. Bers: Riemann's mapping theorem for variable metrics. - Ann. Math. 72, 1960, 385–404.

[A1] Alessandrini, G.: An identification problem for an elliptic equation in two variables. - Ann. Mat. Pura Appl. (4) 145, 1986, 265–296.

[A2] Alessandrini, G.: Critical points of solutions of elliptic equations in two variables. - Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) XIV 2, 1987, 229–256.

[A3] Alessandrini, G.: Critical points of solutions to the p-Laplace equation in dimension two. - Boll. Un. Mat. Ital. A (7) 1, 1987, 239–246.

[AM1] Alessandrini, G., and R. Magnanini: The index of isolated critical points and solutions of elliptic equations in the plane. - Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) XIX 4, 1992, 567–589.

[AM2] Alessandrini, G., and R. Magnanini: Elliptic equations in divergence form, geometric critical points of solutions and Stekloff eigenfunctions. - SIAM J. Math. Anal. 25 (5), 1994, 1259–1268.

[BMN] Bauman, P., A. Marini, and V. Nesi: Univalent solutions of an elliptic system of partial differential equations arising in homogenization. - Preprint, 1998.

[BJS] Bers, L., F. John, and M. Schechter: Partial Differential Equations. - Interscience, New York, 1964.

[BN] Bers, L., and L. Nirenberg: On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications. - In: Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 111–140, Edizioni Cremonese, Roma, 1955.

[BS] Bergman, D.J., and D. Stroud: Physical properties of macroscopically inhomogeneous media. - Solid State Physics 46, 1992, 147–269.

[BI] Bojarski, B., and T. Iwaniec: p-harmonic equation and quasiregular mappings. - Preprint, Universität Bonn, 1983.

[C] Choquet, C.: Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques. - Bull. Sci. Math. 69, 1945, 156–165.

[DH] Duren, P., and W. Hengartner: Harmonic mappings of multiply connected domains. - Pacific J. Math. 180, 1997, 200–220.

[FG] Finn, R., and D. Gilbarg: Asymptotic behavior and uniqueness of plane subsonic flows. - Comm. Pure Appl. Math. 10, 1957, 23–63.

[GT] Gilbarg, G., and N.S. Trudinger: Elliptic Partial Differential Equations of Second Order. - Springer-Verlag, Berlin, 1998.

[HW] Hartman, P., and A. Wintner: On the local behavior of solutions of non-parabolic partial differential equations. - Amer. J. Math. 75, 1953, 449–476.

[HKM] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear Potential Theory of Degenerate Elliptic Equations. - Oxford University Press, Oxford, 1993.

[JM] Jenkins, J.A., and M. Morse: Topological methods on Riemann surfaces. Pseudoharmonic functions. - In: Contributions to the Theory of Riemann Surfaces, Ann. of Math. Stud. 30, Princeton University Press, Princeton, N.J., 1953, pp. 111–139.

[J] Jost, J.: Univalency of harmonic mappings between surfaces. - J. Reine Angew. Math. 324, 1981, 141–153.

[K] Kneser, H.: Lösung der Aufgabe 41. - Jahresber. Deutsche Math.-Verein. 35, 1926, 123–124.

[La] Laugesen, R.S.: Injectivity con fail for higher-dimensional harmonic extentions. - Complex Variables Theory Appl. 28, 1996, 357–369.

[Le] Lewis, J.L.: On critical points of p harmonic functions in the plane. - Electron. J. Differential Equations 3, 1994, 1–4.

[Li] Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. - Nonlinear Anal. 12(11), 1988, 1203–1219.

[LV] Lehto, O., and K.I. Virtanen: Quasikonforme Abbildungen. - Springer-Verlag, Berlin, 1965.

[LK] Levy, O., and R.V. Kohn: Duality relations for non-ohmic composites, with applications to behavior near percolation. - J. Statist. Phys. 90, 1998, 159–189.

[Ly] Lyzzaik, A.: Univalence criteria for harmonic mappings in multiply connected domains. - J. London Math. Soc. 58, 1998, 163–171.

[MZ] Malý, J., and W.P. Ziemer: Fine Regularity of Solutions of Elliptic Partial Differential Equations. - American Mathematical Society, Providence, R.I., 1998.

[Ma] Manfredi, J.J.: p-harmonic functions in the plane. - Proc. Amer. Math. Soc. 103(2), 1988, 473–479.

[Mo] Morse, M.: Topological methods in the theory of functions of a complex variable. - Ann. of Math. Stud. 15, Princeton University Press, Princeton, N.J., 1947.

[R] Radó, T.: Aufgabe 41. - Jahresber. Deutsche Math.-Verein. 35, 1926.

[SY]. Schoen, R., and S.T. Yau: On univalent harmonic mappings between surfaces. - Invent. Math. 44, 1978, 265–278.

[S] Sigalotti, M.: Proprietà geometriche delle soluzioni di equazioni ellittiche degeneri a coefficienti variabili in dimensione due. - Tesi di Laurea, Università degli Studi di Trieste, 1999.

[Ta] Talenti, G.: Equazioni lineari ellittiche in due variabili. - Matematiche (Catania) 21, 1966, 339–376.

[To] Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. - J. Differential Equations 51, 1984, 126–150.

Received 28 September 1999