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ON UNIFORMLY QUASISYMMETRIC GROUPS OF CIRCLE DIFFEOMORPHISMS

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Abstract. This article deals with the conjugacy problem of uniformly quasisymmetric groups of circle homeomorphims to groups of Möbius transformations. We prove that if the involved maps have some degree of regularity and the uniform quasisymmetry can be detected by some natural L^1-cocycle associated to the action, then the conjugacy is, in fact, smooth.

Introduction

Roughly speaking, quasisymmetric homeomorphisms of the circle are those which almost preserve cross-ratios. A subgroup Γ of $\mathrm{Homeo}_+(\mathrm{S}^1)$ is said to be *uniformly quasisymmetric* if all its elements are quasisymmetric and the distortion of cross-ratios is uniformly controlled, independently of the element of the group (see Section 1.1 for more details).

Question 1. Is every uniformly quasisymmetric group of circle homeomorphisms quasisymmetrically conjugate to a group of Möbius transformations?

After some effort by several people, this question has been positively answered by V. Markovic in the recent work [24]. Although both the problem and its answer are very natural, the proof unfortunately appears to be rather difficult. Moreover, it uses some well-known results, as for instance the Convergence Theorem, whose proof is already quite involved.

One of the main difficulties overcome by V. Markovic is the absence of a natural procedure to deal with the problem. Indeed, a difficult result by D. Epstein and V. Markovic [9] shows that there is no canonical *equivariant* extension to the disk of quasisymmetric homeomorphisms of the circle. (The well-known extensions in [1], [7] and [31] fail to be equivariant.) The aim of this work is to present a conceptual approach to Question 1, which unfortunately does not solve it completely, but gives some insight into its nature. We will consider the same question in the smooth case, and in this context we will study the problem of smooth conjugacy.

Question 2. Let Γ be a non metabelian group of $C^{1+\tau}$ circle diffeomorphisms, where $\tau > 0$. If Γ is uniformly quasisymmetric, is Γ necessarily $C^{1+\tau}$ conjugate to a group of Möbius transformations?

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We think that the answer to this question is positive, but this seems not easy to be proved. The main difficulty is that it is not clear how to utilize the smoothness of maps in order to control the distortion of cross-ratios in an optimal way. Recall however the well-known formula

$$\log([a,b,c,d]) = \log\left(\frac{(a-c)(b-d)}{(a-d)(b-c)}\right) = \int_a^b \int_c^d \frac{dx\,dy}{4\sin^2\left(\frac{1}{2}(x-y)\right)},$$

where $a < b < c < d < a$. This equality shows that, for every circle diffeomorphism g,

$$\begin{aligned}\log\left(\frac{[g(a),g(b),g(c),g(d)]}{[a,b,c,d]}\right) \\ = \int_a^b \int_c^d \left[\frac{g'(x)g'(y)}{4\sin^2\left(\frac{1}{2}\big(g(x)-g(y)\big)\right)} - \frac{1}{4\sin^2\left(\frac{1}{2}(x-y)\right)}\right] dx\,dy.\end{aligned} \tag{1}$$

It is then natural to consider the function $c(g)\colon \mathrm{S}^1 \times \mathrm{S}^1 \to \mathbf{R}$ defined by

$$c(g)(x,y) = \frac{g'(x)g'(y)}{4\sin^2\left(\frac{1}{2}\big(g(x)-g(y)\big)\right)} - \frac{1}{4\sin^2\left(\frac{1}{2}(x-y)\right)}.$$

This *Liouville's cocycle* c is a function whose L^1-norm captures in some way the distortion of cross-ratios. Note that a closely related cocycle appears in [26] and [29] in the study of certain cohomological obstructions for group actions on the circle. In another direction, D. Sullivan and S. Nag showed that c is a natural cocycle arising in the analytical theory of quasisymmetric homeomorphisms [25]. They proved in particular that $c(g)$ is the kernel of a continuous linear operator acting on $H^{1/2}(\mathrm{S}^1)$.

Inspired by the afore mentioned works, we will say that a group of circle diffeomorphisms is L^1*-uniformly quasisymmetric* if there exists an upper bound for the L^1-norm of the function $c(g)$ which is independent of the element of the group. Our main result then takes the following form.

Theorem A. *Let* Γ *be a non metabelian subgroup of* $\mathrm{Diff}_+^{r+3}(\mathrm{S}^1)$, *where* $0 \le r \le \infty$. *If* Γ *is* L^1*-uniformly quasisymmetric, then* Γ *is* C^{r+3} *conjugate to a group of Möbius transformations.*

It is very plausible that a refinement of our technique allows to prove a similar statement for L^p-uniformly quasisymmetric groups of $C^{1+\tau}$ diffeomorphisms of the circle when $\tau > 1/p$ (see Section 1.1 for more details). However, we will not pursue this issue here, since we only want to illustrate the idea of our approach and not to obtain sharp results. Moreover, it is quite clear that our method cannot be used to deal with the general quasisymmetric conjugacy problem, which has already been completely solved.

Recall that one of the motivations for Question 1 is the fact that its two-dimensional version (replace quasisymmetric by quasiconformal and the circle by the Riemann sphere) was elegantly solved by D. Sullivan in [34] (see also [35]). His proof uses a simple barycentric argument in order to obtain a conformal invariant structure, and then the Ahlfors–Bers theorem in order to integrate this structure. Our approach to the analogous one-dimensional problem is much inspired by this idea. Indeed, the fact that $c(g)$ is a cocycle suggests the use of some barycenter-type argument. Besides the problem of the smoothness required for the definition of the cocycle, the main difficulty to implement this idea is the absence of an analogue of Ahlfors–Bers' theorem of integrability. However, in the C^3 case it is possible to use Schwarzian derivatives to conclude the existence of an invariant projective structure on the circle, whose integrability is a relatively elementary issue. The use of this last technique is strongly inspired by [11].

The reader could think that, in difference with the general quasisymmetric framework, for our case it is the smoothness of maps which is at the origin of the conjugacy. Nevertheless, this is not at all the case. To illustrate in some way this point, we give in an independent Appendix an example of a group of smooth (namely, real-analytic) diffeomorphisms of the circle whose elements behave individually like Möbius transformations, but which is not topologically conjugate to a group of Möbius transformations. Before stating precisely this result, let us say that a group of circle homeomorphisms is *pseudo-Möbius* if each one of its elements is topologically conjugate to a Möbius transformation, and let us say that such a group is a *Möbius group* if it is topologically conjugate to a subgroup of $\mathrm{PSL}(2,\mathbf{R})$. The following result is a slight improvement of the one obtained by N. Kovačević in [22].

Theorem B. *There exists a group of real-analytic diffeomorphisms of the circle, all whose orbits are dense, which is pseudo-Möbius and non Möbius.*

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1. Some preliminary facts

1.1. Quasisymmetry, Liouville's cocycle and Schwarzian derivative. Throughout this article, we will only deal with orientation-preserving homeomorphisms. Among several equivalent definitions of quasisymmetry, we will only consider the one which is the most appropriated for our study.

Definition 1.1. A circle homeomorphism g is said to be *quasisymmetric* if there exists a constant $M < \infty$ such that for all points $a < b < c < d < a$ on the circle satisfying $[a,b,c,d] = 2$, one has $[g(a),g(b),g(c),g(d)] \leq M$. A subgroup Γ of $\mathrm{Homeo}_+(\mathrm{S}^1)$ is said to be *uniformly quasisymmetric* if all its elements are quasisymmetric with respect to the same constant $M < \infty$.

Equality (1) shows that if g is a circle diffeomorphism satisfying $\|c(g)\| \leq M$, then g is quasisymmetric with respect to the constant $M = 2\exp(\|c(g)\|)$. This shows that if Γ is a L^1-uniformly quasisymmetric group of circle diffeomorphisms (as defined in the Introduction), then Γ is a uniformly quasisymmetric group (as defined above). However, the converse seems to be false, even for real-analytic diffeomorphisms.

For a single diffeomorphism g, the problem of the integrability of $c(g)$ is settled by the following elementary lemma.

Lemma 1.2. *If g is a $C^{2+\tau}$ diffeomorphism of the circle for some $\tau > 0$, then $c(g)$ belongs to $L^1(\mathrm{S}^1 \times \mathrm{S}^1)$.*

Proof. Passing to local coordinates, the problem reduces to proving that

$$\left[\frac{g'(x)g'(y)}{\big(g(x)-g(y)\big)^2} - \frac{1}{(x-y)^2}\right] \in L^1(\mathrm{S}^1 \times \mathrm{S}^1), \tag{2}$$

and obviously the difficulty is to estimate the left-hand expression near the diagonal. To do this, note that a simple development in Taylor series gives, for x near to y,

$$g'(x) = g'(y) + (x-y)g''(y) + o(|x-y|^{1+\tau/2}),$$

and so

$$g'(x)g'(y)(x-y)^2 = (x-y)^2\big(g'(y)\big)^2 + (x-y)^3 g'(y)g''(y) + o(|x-y|^{3+\tau/2}). \tag{3}$$

On the other hand,

$$g(x) - g(y) = (x-y)g'(y) + \tfrac{1}{2}(x-y)^2 g''(y) + o(|x-y|^{2+\tau/2}). \tag{4}$$

Using (3) and (4) one obtains that the left-hand expression in (2) is bounded by another one of the form

$$\frac{o(|x-y|^{3+\tau/2})}{(x-y)^2\big(g(x)-g(y)\big)^2},$$

which has order $o(|x-y|^{\tau/2-1})$ near the diagonal. Since for every $\tau > 0$ the map $(x,y) \mapsto 1/|x-y|^{1-\tau/2}$ belongs to $L^1(\mathrm{S}^1 \times \mathrm{S}^1)$, this proves the lemma. □

Recall that the Schwarzian derivative of a C^3 diffeomorphism $g\colon I \subset \mathbf{R} \to J \subset \mathbf{R}$ is defined by

$$s(g) = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2.$$

This is closely related to Liouville's cocycle, as is illustrated by the well-known formula (which can be obtained as above by a simple development in Taylor series)

$$s(g)(x) = 6 \lim_{y \to x} c(g)(x,y). \tag{5}$$

The validity of (5) implies that if g is a C^{3+r} diffeomorphism, then $c(g)$ is of class C^r on $\mathrm{S}^1 \times \mathrm{S}^1$.

We will denote $\bar{c}(g) = c(g^{-1})$. The reason for introducing the function $\bar{c}$ is that it is a cocycle associated to the left regular representation $\theta\colon \mathrm{Diff}_+^1(\mathrm{S}^1) \to U\big(L^1(\mathrm{S}^1 \times \mathrm{S}^1)\big)$ given by

$$\theta(g^{-1})\xi(x,y) = \xi\big(g(x),g(y)\big)g'(x)g'(y).$$

In other words, the equality

$$\bar{c}(gh) = \bar{c}(g) + \theta(g)\bar{c}(h) \tag{6}$$

holds for every g,h in $\mathrm{Diff}_+^1(\mathrm{S}^1)$. This means that the correspondence $g \mapsto \theta(g)+c(g)$ defines a representation of $\mathrm{Diff}_+^1(\mathrm{S}^1)$ by isometries of the Banach space $L^1(\mathrm{S}^1 \times \mathrm{S}^1)$.

Lemma 1.3. *Let Γ be a L^1-uniformly quasisymmetric group of circle diffeomorphisms. If $\varphi\colon \mathrm{S}^1 \to \mathrm{S}^1$ is a $C^{2+\tau}$ diffeomorphism, then the conjugate group $\varphi\Gamma\varphi^{-1}$ is also L^1-uniformly quasisymmetric.*

Proof. This follows immediately from the cocycle property (6) and Lemma 1.2. Indeed, $\bar{c}(\varphi^{-1}) = -\theta(\varphi^{-1})\bar{c}(\varphi)$, and so

$$\|\bar{c}(\varphi\circ g\circ\varphi^{-1})\| = \|\bar{c}(\varphi) + \theta(\varphi)\bar{c}(g) + \theta(\varphi g)\bar{c}(\varphi^{-1})\| \le 2\|c(\varphi)\| + \|\bar{c}(g)\|.$$

In particular, if $\|c(g)\| \le C$ for all $g \in \Gamma$, then $\|c(\varphi\circ g\circ\varphi^{-1})\| \le C + 2\|c(\varphi)\|$ for all $g \in \Gamma$. □

More generally, for each $p \in]1,\infty[$ we can define the *Liouville's L^p cocycle c_p* by

$$c_p(g)(x,y) = \frac{[g'(x)g'(y)]^{1/p}}{\big|2\sin\big(\frac{1}{2}(g(x)-g(y))\big)\big|^{2/p}} - \frac{1}{\big|2\sin\big(\frac{1}{2}(x-y)\big)\big|^{2/p}}, \quad p \ne 2,$$

$$c_2(g)(x,y) = \frac{\sqrt{g'(x)g'(y)}}{2\sin\big(\frac{1}{2}(g(x)-g(y))\big)} - \frac{1}{2\sin\big(\frac{1}{2}(x-y)\big)}.$$

Note that $c = c_1$. In the L^p setting, $\bar{c}_p(g) = c_p(g^{-1})$ appears as a cocycle associated to the left regular representation θ_p given by

$$\theta_p(g^{-1})\xi(x,y) = \xi\big(g(x),g(y)\big)[g'(x)g'(y)]^{1/p},$$

that is, for $\bar{c}_p(g) = c_p(g^{-1})$ one has

$$\bar{c}_p(gh) = \bar{c}_p(g) + \theta_p(g)\bar{c}_p(h).$$

One easily checks that L^p-uniformly quasisymmetric groups of circle diffeomorphisms are also uniformly quasisymmetric. The proof of the following lemma is analogous to that of Lemmas 1.2 and 1.3, and we will leave it to the reader (see for instance the proof of Proposition 2.1 in [26]).

Lemma 1.4. *If g is an element of $\mathrm{Diff}_+^{1+\tau}(\mathrm{S}^1)$ for some $\tau \in]0,1[$, then $c_p(g)$ belongs to $L^p(\mathrm{S}^1 \times \mathrm{S}^1)$ for all $p > 1/\tau$. Moreover, if Γ is a L^p-uniformly quasisymmetric group and φ is $C^{1+\tau}$ circle diffeomorphism such that $\tau > 1/p$, then the conjugate group $\varphi\Gamma\varphi^{-1}$ is also L^p-uniformly quasisymmetric.*

Among the L^p cocycles for $p > 1$, we will only utilize the cocycle c_2. This is because c_2 takes values on a Hilbert space, and in this context the classical Tits' center lemma [2] implies the following proposition.

Proposition 1.5. *Let $\hat{c}$ be a cocycle with respect to an orthogonal representation $\hat{\theta}$ of a group Γ on a Hilbert space $\mathscr{H}$. If the norm $\|\hat{c}(g)\|$ is bounded by a constant which is independent of g, then there exists a vector $\xi \in \mathscr{H}$ such that $\hat{c}$ is the coboundary associated to ξ, that is $\hat{c}(g) = \xi - \hat{\theta}(g)\xi$ for all $g \in \Gamma$.*

We remark that $\bar{c}_2$ is nothing else than the formal coboundary of the function (which does not belong to L^2)

$$(x,y) \longmapsto -\frac{1}{2\sin\big(\frac{1}{2}(x-y)\big)}.$$

Quite naturally, the cocycle c_2 has better regularity properties near the diagonal than c. Indeed, if g is of class C^2, then one easily checks that c_2 extends continuously to $\mathrm{S}^1 \times \mathrm{S}^1$ as being identically zero on the diagonal. To ensure the existence (and continuity) of $\partial_1 c(g)(x,x)$ and $\partial_2 c(g)(x,x)$ (a fact that will be essential in what follows), the Taylor development

$$c_2(g)(x,y) = \frac{g'''(y)}{12}(x-y) + o(|x-y|) \tag{7}$$

shows that a C^3 regularity hypothesis for g suffices. We leave to the reader the easy verification of (7).

1.2. Projective structures. A one-dimensional manifold X is endowed with a projective structure when we fix a system of local charts $\varphi_i\colon I_i \subset \mathrm{X} \to \mathbf{R}$ such that the change of coordinate maps $\varphi_i \circ \varphi_j^{-1}$ are (restrictions of) Möbius transformations. If $g\colon \mathrm{X}_1 \to \mathrm{X}_2$ is a C^3 local diffeomorphism between one-dimensional projective manifolds, then the quadratic differential form $s(g)\,dx^2$ is well defined on X_1, that is it does not depend on projective charts. The following properties are satisfied:

(i) One has $s(g)\,dx^2 = 0$ if and only if g is a projective diffeomorphism. This follows from the fact that the only local diffeomorphisms with zero Schwarzian derivative are the restrictions of Möbius transformations.

(ii) If $g\colon \mathrm{X}_1 \to \mathrm{X}_2$ and $h\colon \mathrm{X}_2 \to \mathrm{X}_3$ are C^3 local diffeomorphisms between projective manifolds, then

$$s(h \circ g)\,dx^2 = g^*\big(s(h)dx^2\big) + s(g)\,dx^2. \tag{8}$$

This can be obtained by a straightforward computation, but it can also be easily deduced from (5) and (6).

If X is a one-dimensional projective manifold and s is a quadratic differential form defined on it which is of class C^r for some $0 \le r \le \infty$, then the equation $s(g)\,dx^2 = s$ has local solutions of class C^{r+3}, and two solutions differ by (the restriction of) a Möbius transformation. So, there is a natural bijection between projective structures of class C^{r+3} on X and quadratic differential forms of class C^r.

By (ii), a diffeomorphism $g\colon \mathrm{X} \to \mathrm{X}$ preserves the projective structure given by a quadratic form $s = \zeta\, dx^2$ if and only if $s = g^*(s) + s(g)\,dx^2$, which by (8) is equivalent in projective coordinates to

$$\zeta = (g')^2(\zeta \circ g) + s(g). \tag{9}$$

Example 1.6. Let us consider the local charts φ_1 and φ_2 on S^1 given by

$$\varphi_1(x) = \tan\big(\tfrac{1}{2}x\big), \quad x \ne \pi; \qquad \varphi_2(x) = \tan\big(\tfrac{1}{2}x + \tfrac{1}{4}\pi\big), \quad x \ne \tfrac{1}{2}\pi.$$

The change of coordinates map is then given by

$$\varphi_2 \circ \varphi_1^{-1}(x) = \frac{1+x}{1-x},$$

which proves that the charts φ_1 and φ_2 define a projective structure on S^1. This structure is invariant by the action of $\mathrm{PSL}(2,\mathbf{R})$. In fact, the group of diffeomorphisms which preserve this structure coincides with the Möbius group.

The following classical result will be essential for our proof. It was first obtained by N. Kuiper in [23] with a little mistake in the proof, which was corrected by W. Goldman in [15], [16]; see also [32].

Theorem (Kuiper–Goldman). *If the group of automorphisms of a C^r projective structure on the circle is non metabelian, then it is C^r conjugate to some finite covering of* $\mathrm{PSL}(2,\mathbf{R})$.

2. Smooth conjugacy

2.1. An outline of the proof. Let Γ be a group of $C^{2+\tau}$ diffeomorphisms of the circle. Recall the cocycle c_2 which has been defined by

$$c_2(g)(x,y) = \frac{\sqrt{g'(x)g'(y)}}{2\sin\big(\frac{1}{2}(g(x)-g(y))\big)} - \frac{1}{2\sin\big(\frac{1}{2}(x-y)\big)}.$$

For every circle diffeomorphism g and all x, y in S^1 one has

$$|c_2(g)(x,y)|^2 \le |c(g)(x,y)|.$$

From this one concludes that if Γ is L^1-uniformly quasisymmetric, then the restriction to Γ of the cocycle c_2 is uniformly bounded. By Proposition 1.5, there exists a function $\xi \in L^2(\mathrm{S}^1 \times \mathrm{S}^1)$ such that $\bar{c}_2(g) = \xi - \theta_2(g)\xi$ for all $g \in \Gamma$.

Let us assume that ξ has continuous ∂_1 and ∂_2 derivatives on a neighborhood of the diagonal Δ in $\mathrm{S}^1 \times \mathrm{S}^1$. In this situation the following proposition holds.

Proposition 2.1. *Let* Γ *be a subgroup of* $\mathrm{Diff}_+^{3+r}(\mathrm{S}^1)$, *where* $r \geq 0$. *Let us suppose that there exists a measurable and square integrable function* $\xi\colon \mathrm{S}^1 \times \mathrm{S}^1 \to \mathbf{R}$ *such that* $\bar{c}_2(g) = \xi - \theta_2(g)\xi$ *for all* $g \in \Gamma$. *If* ξ *is continuous and has continuous* ∂_1 *and* ∂_2 *derivatives near the diagonal, then there exists a probability measure on* S^1 *which is invariant by* Γ *and whose density function is of class* C^{r+1}, *or there exists a projective structure of class* C^3 *on* S^1 *which is invariant by* Γ.

In the non metabelian case, the existence of an invariant probability measure is not possible.

Lemma 2.2. *Let* Γ *be a uniformly quasisymmetric group of circle homeomorphisms. If* Γ *is non metabelian, then it cannot preserve any probability measure on* S^1.

So, if Γ is a non metabelian L^1-uniformly quasisymmetric group of C^{r+3} circle diffeomorphisms such that the corresponding function ξ has continuous ∂_1 and ∂_2 derivatives near the diagonal, then Γ preserves a C^3 projective structure on S^1. Again since Γ is supposed to be non metabelian, using Lemma 2.2 and Kuiper–Goldman's theorem one can conclude that this projective structure is C^3 conjugate to the canonical one. By the main result of [14], this conjugacy is in fact of class C^{r+3}, finishing the proof of Theorem A.

We close this section with the proof of Proposition 2.1. In the next section we will check that, under our hypothesis, the regularity properties for the kernel function ξ are satisfied.

Proof of Proposition 2.1. Note that the hypothesis implies that, for all $g \in \Gamma$ and a.e. $(x,y) \in \mathrm{S}^1 \times \mathrm{S}^1$,

$$\left[\frac{1}{2\sin\big(\frac{1}{2}\big(g(x)-g(y)\big)\big)} + \xi\big(g(x),g(y)\big)\right] g'(x)g'(y) = \frac{1}{2\sin\big(\frac{1}{2}(x-y)\big)} + \xi(x,y).$$

This means that the measure ν on $\mathrm{S}^1 \times \mathrm{S}^1 \setminus \Delta$ with density function

$$(x,y) \mapsto \frac{1}{2\sin\big(\frac{1}{2}(x-y)\big)} + \xi(x,y)$$

is Γ-invariant. The idea of the proof of the proposition is very simple: the infinitesimal change of ν near the diagonal is a quadratic differential form, which has to be invariant since the measure ν is invariant.

With respect to the canonical projective structure on S^1, the density of the measure ν has the local form

$$\left[\frac{1}{x-y} + N(x,y)\right]^2 dx\,dy,$$

where N is a measurable function which is continuous and has continuous ∂_1 and ∂_2 derivatives on a neighborhood of the diagonal in $\mathbf{R}\times\mathbf{R}$. Changing ξ by the function $(x,y)\mapsto \xi(x,y)+\xi(y,x)$ if necessary, we may assume that $N(x,y)=N(y,x)$ for all x,y in $\mathbf{R}$. By (9), in order to obtain a C^r invariant projective structure we have to prove that there exists a continuous function $\zeta\colon \mathbf{R}\to\mathbf{R}$ such that, for all $g\in\Gamma$,

$$\zeta=(g')^2(\zeta\circ g)+s(g).$$

We will prove that this equality holds for $\zeta(x)=12\partial_1 N(x,x)$ when the function ξ is identically zero on the diagonal. If this is not the case, we will show that the (finite total mass) measure on S^1 whose density function is given by $x\mapsto|\xi(x,x)|$ is invariant by Γ.

By hypothesis, for all $g\in\Gamma$ and all $x\neq y$ in $\mathbf{R}$ one has

$$g'(x)g'(y)\left[\frac{1}{g(x)-g(y)}+N\big(g(x),g(y)\big)\right]^2=\left[\frac{1}{x-y}+N\big(g(x),g(y)\big)\right]^2,$$

which gives

$$\begin{aligned}(10)\qquad g'(x)g'(y)(x-y)^2\big[1+\big(g(x)-g(y)\big)N\big(g(x),g(y)\big)\big]^2&\\ =[1+(x-y)N(x,y)]^2\big(g(x)-g(y)\big)^2.&\end{aligned}$$

Note that for x near y one has

$$g'(x)=g'(y)+(x-y)g''(y)+o(|x-y|),$$

and so

$$(11)\quad g'(x)g'(y)(x-y)^2=(x-y)^2\big(g'(y)\big)^2+(x-y)^3g'(y)g''(y)+o(|x-y|^3).$$

On the other hand,

$$g(x)-g(y)=(x-y)g'(y)+\tfrac{1}{2}(x-y)^2g''(y)+o(|x-y|^2),$$

$$N\big(g(x),g(y)\big)=N\big(g(y),g(y)\big)+\partial_1N\big(g(y),g(y)\big)g'(y)(x-y)+o(|x-y|),$$

and so

$$(12)\quad \big[1+\big(g(x)-g(y)\big)N\big(g(x),g(y)\big)\big]^2=1+(x-y)g'(y)N\big(g(y),g(y)\big)+o(|x-y|).$$

From (11) and (12) one verifies that the left-hand member of equality (10) is equal to

$$(13)\quad (x-y)^2\big(g'(y)\big)^2+(x-y)^3\big[g'(y)g''(y)+2\big(g'(y)\big)^3N\big(g(y),g(y)\big)\big]+o(|x-y|^3).$$

From the equalities

$$[1+(x-y)N(x,y)]^2 = [1+(x-y)N(y,y)+o(|x-y|)]^2$$
$$= 1+2(x-y)N(y,y)+o(|x-y|),$$

$$\big(g(x)-g(y)\big)^2 = (x-y)^2\big(g'(y)\big)^2 + (x-y)^3 g'(y)g''(y) + o(|x-y|^3),$$

one concludes that the right-hand member of equality (10) is equal to

$$(x-y)^2\big(g'(y)\big)^2 + (x-y)^3\big[g'(y)g''(y) + 2\big(g'(y)\big)^2 N(y,y)\big] + o(|x-y|^3). \tag{14}$$

Thus, replacing (13) and (14) in (10), and then identifying the coefficients of $(x-y)^3$, one concludes that

$$2\big(g'(y)\big)^2 N(y,y) = 2\big(g'(y)\big)^3 N\big(g(y),g(y)\big),$$

that is

$$N\big(g(y),g(y)\big)g'(y) = N(y,y).$$

This shows that the measure on S^1 with density function $x \mapsto |\xi(x,x)|$ is invariant by Γ. The total mass of S^1 by this measure is finite, but it can be equal to zero.

Let us suppose in what follows that this measure is trivial. As in the first part of the proof, we develop both sides of equality (10) in Taylor's series, but this time until the fourth order term. For the coefficient of $(x-y)^4$ in the left-hand member of (10) one finds

$$\tfrac{1}{2}\big(g'(y)g'''(y)\big) + 2\big(g'(y)\big)^4 \partial_1 N\big(g(y),g(y)\big),$$

whereas for the coefficient of $(x-y)^4$ of the right-hand member one finds

$$\tfrac{1}{4}\big(g''(y)\big)^2 + \tfrac{1}{3}g'(y)g'''(y) + 2\big(g'(y)\big)^2 \partial_1 N(y,y).$$

Thus,

$$\tfrac{1}{2}g'(y)g'''(y) + 2\big(g'(y)\big)^4 \partial_1 N\big(g(y),g(y)\big)$$
$$= \tfrac{1}{4}(g''(y))^2 + \tfrac{1}{3}g'(y)g'''(y) + 2\big(g'(y)\big)^2 \partial_1 N(y,y),$$

that is,

$$\frac{g'''(y)}{g'(y)} - \frac{3}{2}\left(\frac{g''(y)}{g'(y)}\right)^2 = 12\partial_1 N(y,y) - 12\big(g'(y)\big)^2 \partial_1 N\big(g(y),g(y)\big),$$

which is the desired equality. ▫

2.2. The regularity of the kernel. It is not difficult to prove that every uniformly quasisymmetric group of circle homeomorphisms satisfies the convergence property, and so it is topologically conjugate to a Möbius group [5], [10]. (Note that Lemma 2.2 follows directly from this.) Since we want to avoid the use of this deep and difficult result, we will need to obtain *a priori* some simple but useful topological information. The following lemma is well known, and we include a proof only for the convenience of the reader.

Lemma 2.3. *Let g be a circle homeomorphism such that the group generated by g is uniformly quasisymmetric. If g fixes three points of the circle, then g coincides with the identity. Moreover, if g fixes only two points, then one of them is* (*topologically*) *attracting and the other is* (*topologically*) *repelling.*

Proof. If g fixes more than two points then, passing to local projective coordinates, we are reduced to the case where g is a homeomorphism of the real line having at least two fixed points. Let us suppose that there exists a connected component $]a,b[$ of the complement of the set of fixed points of g such that both a and b are real numbers, and let $c = \frac{1}{2}(a+b)$. Changing g by g^{-1} if necessary, we may assume that $g(x) < x$ for all $x \in]a,b[$. The point of $g^n(c)$ goes to a, and so the value of

$$\frac{g^n(b) - g^n(c)}{g^n(c) - g^n(a)}$$

diverges as n goes to infinity, contradicting the quasisymmetry hypothesis. Let us now suppose that there is no such connected component $]a,b[$. Again, changing g by g^{-1} if necessary, we may assume that there exists $a \in \mathbf{R}$ such that either $g(x) = x$ for $x \le a$ and $g(x) > x$ for $x > a$, or $g(x) = x$ for $x \ge a$ and $g(x) > x$ for $x < a$. Both cases being analogous, let us only consider the first one. For $t > 0$ the point $g^n(a+t)$ goes to the infinity, and $g^n(a-t) = a-t$. Thus, the value of

$$\frac{g^n(a+t) - g^n(a)}{g^n(a) - g^n(a-t)}$$

diverges as n goes to infinity, which again contradicts the quasisymmetry hypothesis. The case of two fixed points can be treated by similar arguments and we leave it to the reader. □

Proof of Lemma 2.2. We will prove more generally that if Γ is a group of circle homeomorphisms whose non trivial elements fix at most two points and which preserves a probability measure on S^1, then Γ is metabelian. Indeed, if the invariant probability measure has no atomic part, then its support is a closed set invariant by the action of Γ. It is then easy to see that the derived group $\Gamma' = [\Gamma, \Gamma]$ fixes punctually this set. By our hypothesis, Γ' is trivial, and so Γ is Abelian (and in fact semiconjugate to a group of rotations). On the other hand, if we assume that Γ preserves an atomic probability measure on S^1, then the atoms of maximal mass form a finite set F which is invariant by Γ. Thus, each element of a finite index subgroup Γ_0 of Γ fixes every point in F. If F contains more than two points then by hypothesis Γ_0 is trivial, and so Γ is finite (and indeed conjugate to a finite group of rotations). If F contains one or two points, then the group Γ_0 fixes globally one point. By Holder's and Solodov's theorems (see the Appendix), this implies that Γ is metabelian (and topologically semiconjugate to a group of affine transformations when uniformly quasisymmetric). □

It is not difficult to verify that uniformly quasisymmetric metabelian groups of circle homeomorphisms are not only semiconjugate but also conjugate to groups of rotations or affine transformations. However, we will not use this fact in the sequel.

Lemma 2.4. *Let $g\colon \mathrm{S}^1 \to \mathrm{S}^1$ be a C^{r+3} diffeomorphism, where $0 \leq r \leq \infty$. Suppose that a is a hyperbolic fixed point of g, and that g has exactly two fixed points a and b. Suppose also that the infinite cyclic group generated by g is L^2-uniformly quasisymmetric, and let ξ be a function in $L^2(\mathrm{S}^1 \times \mathrm{S}^1)$ such that $\bar{c}_2(g) = \xi - \theta_2(g)\xi$. Then ξ coincides on almost every point of a neighborhood of the set $\Delta \setminus \{(a,a),(b,b)\}$ with a function having continuous ∂_1 and ∂_2 derivatives on the diagonal.*

Proof. Using Yoccoz' version of Sternberg's linearization theorem [37], we can conjugate g near the hyperbolic fixed point to the corresponding linear germ. Since linear maps are projective, after conjugacy one has $\bar{c}_2(g)(x,y) = 0$ when x and y are simultaneously near to a. In what follows we will assume that we have performed this conjugacy, and we will fix $a' \in]a,b[$ such that (after conjugacy)

$$\bar{c}_2(g)(x,y) = 0 \quad \text{for all } (x,y) \in [a,a']^2.$$

Changing g by its inverse if necessary, we may assume that $g(x) > x$ for all $x \in]a,b[$.

By Lemma 1.4, the group generated by (the conjugate of) g is also L^2-uniformly quasisymmetric. Let us fix a point $c \in]a,b[$, and let

$$\Delta(c) = \bigcup_{n \in \mathbf{Z}} [g^n(c), g^{n+1}(c)]^2.$$

Note that $\Delta(c)$ is invariant by g. Let us define $\bar{\xi}\colon \Delta(c) \to \mathbf{R}$ by

$$\bar{\xi} = \sum_{i \geq 0} \theta_2(g^i)\bar{c}_2(g).$$

Note that the sum above is well defined, since it involves only a finite number of nonzero terms. In particular, it defines a function with continuous ∂_1 and ∂_2 derivatives on the interior of $\Delta(c)$. Moreover, one easily checks that $\bar{c}_2(g)(x,y) = \bar{\xi}(x,y) - \theta_2(g)\bar{\xi}(x,y)$ for all $(x,y) \in \Delta(c)$.

Now taking into account our hypothesis we conclude that the equality

$$\xi - \theta_2(g)\xi = \bar{\xi} - \theta_2(\bar{\xi})$$

holds for a.e. point (x, y) in $\Delta(c)$. We claim that this implies that ξ and $\bar{\xi}$ coincide a.e. on $\Delta(c)$. Indeed, if not then $|\xi - \bar{\xi}|^2$ would be the density of a non trivial measure on $\Delta(c)$ which is invariant by the diagonal action of g. The total mass of the set $\Delta(c)$ by this measure is finite (since ξ and $\bar{\xi}$ are square integrable). However, this is absurd, since the only finite total mass measures on the closure of $\Delta(c)$ which are invariant by g are the linear combinations of the Dirac measures supported on the fixed points (a, a) and (b, b).

Thus, the function ξ coincides on $\Delta(c)$ with the function $\bar{\xi}$. Changing c by some point $c' \in]c, g(c)[$, we conclude that ξ can be taken as a function with continuous ∂_1 and ∂_2 derivatives on a neighborhood of the diagonal of $]a, b[\times]a, b[$. The proof is finished by looking at the interval $]b, a[$ instead of $]a, b[$, and using the same arguments as before. □

Now let Γ be a L^1-uniformly quasisymmetric group of C^{r+3} diffeomorphisms of the circle. By Section 2.1 we know that there exist a function $\xi \in L^2(\mathrm{S}^1 \times \mathrm{S}^1)$ such that $\bar{c}_2(g) = \xi - \theta_2(g)\xi$ for all $g \in \Gamma$. By Sacksteder's theorem (see [33] and p. 11 of [8]; see also [6]), if Γ is non metabelian then is contains an element h having a hyperbolic fixed point a. Lemma 2.3 implies that h has exactly one fixed point b in $\mathrm{S}^1 \setminus \{a\}$. This allows to apply Lemma 2.4, and so we conclude that the function ξ can be supposed to have continuous ∂_1 and ∂_2 derivatives on $\Delta \setminus \{(a, a), (b, b)\}$. The cocycle identity (6) implies that the set of points with this property is invariant by Γ. So, conjugating by elements sending a and b into points in $\mathrm{S}^1 \setminus \{a, b\}$ (which exist since there is no finite orbit for Γ), one concludes that ξ can be taken as a map with continuous ∂_1 and ∂_2 derivatives on the whole diagonal $\Delta \subset \mathrm{S}^1 \times \mathrm{S}^1$. We are then under the hypothesis of Proposition 2.1, which allows to finish the proof of Theorem A as in Section 2.1.

2.3. Some final comments. The metabelian hypothesis for Theorem A is quite natural. For instance, it is very likely that, for Abelian groups, smooth conjugacy is prevented in many cases by problems related to small denominators (for free actions) [17], or Mather's invariant (for actions with a global fixed point) [37]. In the general metabelian case, there seems also to be some obstructions. Indeed, by combining Lemma 2.3 with the results of [28], one easily concludes that uniformly quasisymmetric groups of C^2 circle diffeomorphisms are topologically conjugate to groups of affine transformations. However, it is well known that in this setting there are (at least local) obstructions to smoothing [36]. Note that in this context, even the problem of the quasisymmetric conjugacy is solved [19] by arguments which are different (and simpler) from those used for the general case [24].

A second remark concerning the statement of Theorem A is related to the smoothness hypothesis. It is clear that the C^3 hypothesis is essential for the use of Schwarzian derivative. As we already remarked, it is very plausible to obtain an analogous version of Theorem A in class $C^{1+\tau}$ for any $\tau > 0$. Nevertheless, it should be said that C^1 conjugacy does not necessarily exist for uniformly quasisymmetric groups of C^1 diffeomorphisms. This was already remarked by É. Ghys in a slightly different context ([11, p. 181]). Finally, let us point out that a real-analytic version of the theorem still holds. This can be checked by a careful analysis of the preceding proof, but it becomes more transparent by putting together its C^∞ version with the real-analytic case of the main result of [14].

We finish this section with a result which is related to the preceding proof. It turns out that Liouville's cocycle is very interesting from a dynamical point of view. This has been already remarked in [30], where the author proved a version of the proposition below for circle diffeomorphisms. Following [37], for $r \geq 0$ let us denote by $\mathrm{Diff}_+^{r,\Delta}([0,1])$ the space of C^r diffeomorphisms of the interval without fixed points on $]0,1[$.

Proposition 2.5. *If $r \geq 3$, then for a generic element g in $\mathrm{Diff}_+^{r,\Delta}([0,1])$ the restriction of Liouville's cocycle to the infinite cyclic group generated by g is non cohomologically trivial.*

Proof. The germs at 0 and 1 of a generic element in $\mathrm{Diff}_+^{r,\Delta}([0,1])$ are hyperbolic. We claim that if the restriction of Liouville's cocycle to the infinite cyclic group generated by such a diffeomorphism g is uniformly bounded, then g preserves a C^r projective structure on $[0,1]$. Before proving this, let us finish our argument. Integrating the invariant projective structure, one concludes that g is C^r conjugate to (the restriction of) a Möbius transformation. In particular, g is contained in a flow of C^r diffeomorphisms. However, it is well known that generic diffeomorphisms are not contained in such a flow [22], [37].

Thus, in order to complete the proof, we have to justify our assertion concerning the existence of a C^r invariant projective structure. To do this, let us first conjugate g by a C^r diffeomorphism in such a way it becomes linear near the end points $a = 0$ and $b = 1$, and let us fix $a' < b'$ in $]a, b[$ such that (after conjugacy) one has $c(g)(x, y) = 0$ for all $(x, y) \in [a, a']^2 \cup [b', b]^2$. We claim that the hypothesis of boundedness of $\|c(g^n)\|$ then implies that, for all $m \in \mathbf{Z}$ and all x, y contained in the interval $[g^m(a'), g^{m+1}(a')]$, one has

$$\sum_{n \in \mathbf{Z}} \theta(g^n)\bar{c}(g)(x, y) = 0. \tag{15}$$

To verify this equality let us denote

$$\sigma(g)(x, y) = \sum_{n \in \mathbf{Z}} \theta(g^n)\bar{c}(g)(x, y).$$

Note that the sum above is well defined, since it involves only a finite number of nonzero terms. Moreover, using (6) one easily checks that $\sigma(g)(x, y) =$

$\theta(g^i)\sigma(g)(x,y)$ for all (x,y) and all $i \in \mathbf{Z}$. Thus, in order to verify equality (15), it suffices to prove it when x and y both belong to $[a', g(a')]$. For each $N \in \mathbf{N}$ denote

$$\sigma_N(g)(x,y) = \bar{c}\big(g^N)(x,y)\big) = \sum_{n=0}^{N-1} \theta(g^n)\bar{c}(g)(x,y).$$

We leave to the reader to verify that, if $0 \leq i < N$ and $(x,y) \in [g^i(a'), g^{i+1}(a')]^2$, then

$$\sigma_N(g)(x,y) = \sigma(g)(x,y),$$

and thus

$$\begin{aligned}\int_{g^i(a')}^{g^{i+1}(a')} \int_{g^i(a')}^{g^{i+1}(a')} |\sigma_N(g)(x,y)|\, dx\, dy &= \int_{g^i(a')}^{g^{i+1}(a')} \int_{g^i(a')}^{g^{i+1}(a')} |\sigma(g)(x,y)|\, dx\, dy \\ &= \int_{g^i(a')}^{g^{i+1}(a')} \int_{g^i(a')}^{g^{i+1}(a')} |\theta(g^i)\sigma(g)(x,y)|\, dx\, dy \\ &= \int_{a'}^{g(a')} \int_{a'}^{g(a')} |\sigma(g)(x,y)|\, dx\, dy.\end{aligned}$$

Denoting by I the value of the last expression, we have

$$\begin{aligned}\int_a^b \int_a^b |\bar{c}(g^N)(x,y)|\, dx\, dy &= \int_a^b \int_a^b |\sigma_N(g)(x,y)|\, dx\, dy \\ &\geq \sum_{i=0}^{N-1} \int_{g^i(a')}^{g^{i+1}(a')} \int_{g^i(a')}^{g^{i+1}(a')} |\sigma_N(g)(x,y)|\, dx\, dy = NI.\end{aligned}$$

If $\sigma(g)$ is not identically zero on $[a', g(a')]^2$ then $I > 0$, and so $\|c(g^{-N})\|$ goes to infinity with N. In particular, the restriction of c to the group generated by g is non uniformly bounded, which contradicts our hypothesis. [1]

Now we localize equality (15) on the diagonal: for all $x \in \mathrm{S}^1$ one has

$$\sum_{n\in\mathbf{Z}} (g^n)^* s(g)(x) = 0. \tag{16}$$

We claim that this implies that the cohomological equation (9) has a C^{r-3} solution ζ on $[a,b]$ such that $\zeta(x) = 0$ for all $x \in [a,a'] \cup [g^k(a'), b]$. To prove this, let us fix $k \in \mathbf{N}$ such that $g^k(a') > b'$. Define inductively ζ: $[a,b] \to \mathbf{R}$ by

$$\zeta(x) = 0 \quad \text{for all } x \in [a,a'].$$

1 It is interesting to remark that the proof shows that if $\|c(g^N)\|$ growths sublinearly, then it is in fact bounded. This should be compared with [30].

Now assuming that ζ has been already defined on $[a, g^{i-1}(a')]$, let

$$\zeta\big(g(x)\big) = \frac{1}{\big(g'(x)\big)^2}\big[\zeta(x) - s(g)(x)\big].$$

We claim that, according to this definition, $\zeta(x) = 0$ for all $x > b'$. Indeed, one easily checks by induction that, for all $i \geq 0$,

$$\zeta\big(g^i(x)\big) = \frac{1}{\big((g^i)'(x)\big)^2}\Big[\zeta(x) - \sum_{n=0}^{i-1}(g^n)^* s(g)(x)\Big].$$

So, if $x \in [g^k(a'), g^{k+1}(a')]$ then

$$\begin{aligned}\zeta(x) &= \zeta\big(g^{k+1}\big(g^{-(k+1)}(x)\big)\big)\\ &= \frac{1}{\big((g^{k+1})'\big(g^{-(k+1)}(x)\big)\big)^2}\Big[\zeta\big(g^{-(k+1)}(x)\big) - \sum_{n=0}^{k}(g^n)^* s(g)\big(g^{-(k+1)}(x)\big)\Big].\end{aligned}$$

But $\zeta\big(g^{-(k+1)}(x)\big) = 0$ since $g^{-(k+1)}(x) < a'$, and moreover

$$\sum_{n=0}^{k}(g^n)^* s(g)\big(g^{-(k+1)}(x)\big) = 0$$

by (16). One thus concludes that $\zeta(x) = 0$ for all $x \in [g^k(a'), g^{k+1}(a')]$. From this one easily checks by induction that $\zeta(x) = 0$ for all $x > g^{k+1}(a')$, and this finishes the proof. □

It would be interesting to study the same kind of phenomena in other contexts. For instance, one could restrict to diffeomorphisms which are tangent to the identity at the end points. On the other hand, it is natural to investigate what happens in lower differentiability classes for the corresponding Liouville's L^p cocycle. Finally, the study of c_2 in reduced cohomology remains a major problem (see however [30] for the case of the circle).

3. Appendix

A theorem essentially due to Hölder says that every group of homeomorphisms of the real line (or of the circle) whose non trivial elements do not fix any point is isomorphic to a group of translations (or of rotations), and the corresponding actions are topologically semiconjugate. In 1991, V. Solodov proved an analogous result for groups of homeomorphisms of the real line without global fixed points and whose non trivial elements fix at most one point: such a group is necessarily isomorphic to a subgroup of the affine group, and the corresponding actions are semiconjugate. The reader is referred to [13] for a nice exposition of these two results.

Motivated by Hölder and Solodov theorems, it was natural to ask if an analogous statement is true for subgroups of $\mathrm{Homeo}_+(\mathrm{S}^1)$ having dense orbits and whose non trivial elements have at most two fixed points, taking Möbius groups as universal models. The (negative) answer to this question was given by N. Kovačević in [22]. However, the maps involved in her nice examples do not satisfy *a priori* any regularity property. The aim of this Appendix is to prove that, by refining the technique of construction of one of her examples, it is possible to obtain a group of *real-analytic diffeomorphisms* of the circle whose orbits are dense, which is pseudo-Möbius and non Möbius.

Let us start our construction by considering a real-analytic diffeomorphism $H\colon \mathbf{R} \to \mathbf{R}$ satisfying $H\big(x+\frac{1}{2}\big) = H(x)+1$ for all $x \in \mathbf{R}$, $\mathrm{Fix}\,(H) \cap \big[\frac{1}{2},1\big] = \big\{\frac{3}{4},\frac{7}{8},1\big\}$, and

$$H'(x) > H'(y) \quad \text{for all } x \in \left]\tfrac{1}{2},\tfrac{3}{4}\right[\text{ and all } y \in \left]\tfrac{3}{4},1\right[. \tag{17}$$

To construct H one can start by considering a real-analytic diffeomorphism $\widehat{H}\colon \mathbf{R} \to \mathbf{R}$ satisfying $H(x+1) = H(x)+1$ for all $x \in \mathbf{R}$, $\mathrm{Fix}\,(\widehat{H}) = \mathbf{Z}$, $\widehat{H}\left(\frac{1}{2}\right) = \frac{3}{4}$, $\widehat{H}\left(\frac{3}{4}\right) = \frac{7}{8}$, and such that $\widehat{H}'(x) > \widehat{H}'(y)$ for all $x \in \left]0,\frac{1}{2}\right[$ and all $y \in \left]\frac{1}{2},1\right[$. (It is easy to verify the existence of such a diffeomorphism.) Then one defines H by $H(x) = \widehat{H}(2x-1)$. The map H induces a degree 2 map of the circle onto itself, which we will still denote by H. Let $A_2 = \left]\frac{1}{4},\frac{3}{8}\right[$, $B_2 = \left]\frac{3}{8},\frac{1}{2}\right[$, $C_2 = \left]\frac{1}{4},\frac{1}{2}\right[$, $A_1 = \left]\frac{3}{4},\frac{7}{8}\right[$, $B_1 = \left]\frac{7}{8},1\right[$ and $C_1 = \left]\frac{3}{4},1\right[$.

Let us now define $g_1\colon [0,1] \to [0,1]$ by $g_1(x) = y$ if $H(x) = H(y)$ and $x \neq y$. It is clear that g_1 coincides with the Euclidean order 2 rotation (see Figure 1). In general, for each $n \in \mathbf{N}$ let us consider the degree 2^n map $H^n\colon \mathrm{S}^1 \to \mathrm{S}^1$, and let us define $g_n\colon \mathrm{S}^1 \to \mathrm{S}^1$ by $g_n(x) = y$ if $H^n(x) = H^n(y)$ and $H^n(x) \neq H^n(y')$ for all $y' \in]x,y[$, where $]x,y[$ is the open interval joining the points x and y (with respect to the canonical orientation of the circle). Note that $g_n^2 = g_{n-1}$ for all $n \geq 2$.

The group Γ_0 generated by the elements in $\{g_n, n \in \mathbf{N}\}$ is a group of real-analytic diffeomorphisms of the circle which is Abelian and torsion. It is indeed an example of a pseudo-Möbius group which is non Möbius and which admits an exceptional minimal set, namely

$$\mathrm{K} = \mathrm{S}^1 \setminus \bigcup_{n \in \mathbf{N}} \bigcup_{i=0}^{2^n-1} g_n^i(C_1).$$

This example, essentially due to M. Hirsch [20], is especially interesting by the fact that, if the map H is well chosen, then the Lebesgue measure of K is positive. (This should be compared with Remark 4.6 in [27].)

Remark 3.1. The lifting to the real line of the group constructed above is a group of real-analytic diffeomorphisms which is semiconjugate (and non conjugate) to a group of translations. Adding the diffeomorphism H, one obtains a subgroup of $\mathrm{Diff}_+^w(\mathbf{R})$ which is semiconjugate (and non conjugate) to a non Abelian subgroup of the affine group.

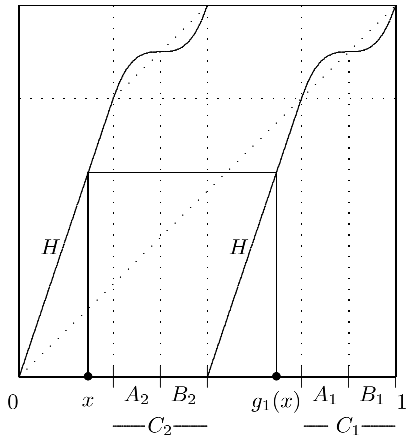

Figure 1.

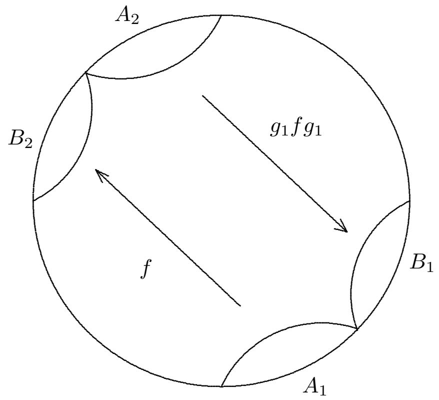

Figure 2.

Now let us fix a real-analytic diffeomorphism $f\colon \mathrm{S}^1 \to \mathrm{S}^1$ having (exactly) two fixed points and such that $f(A_1) = \mathrm{S}^1 \setminus \overline{B_2}$, with $f'(x) > 1$ for all $x \in A_1$ and $f'(x) < 1$ for all $x \in \mathrm{S}^1 \setminus \overline{A_1}$ (see Figure 2). It is easy to see that one can take f as a genuine hyperbolic Möbius transformation.

Let Γ be the group generated by f and the g_n. Using a Klein type ping-pong argument, it is easy to prove that Γ is the (non Abelian) free product between Γ_0 and the infinite cyclic group generated by f. It is very interesting to remark that Γ is discrete with respect to the topology induced from $\mathrm{Homeo}_+(\mathrm{S}^1)$.

In what follows we will give a short and almost self-contained proof of the fact that Γ is pseudo-Möbius, that all its orbits are dense, and that Γ is non Möbius. The proof consists of several steps.

Claim 1. *All the orbits by* Γ *are dense.*

This is delicate to prove, and it is, in fact, our main improvement to Kovačević's construction. Let us remark that in [22] the problem of density of the orbits is settled by collapsing the intervals of the complement of an eventual exceptional minimal set. However, this argument cannot be applied if we want to obtain smooth maps. Our argument uses a result due to G. Hector, which establishes that if a group of real-analytic diffeomorphisms of the circle admits an exceptional minimal set, then the stabilizers of points are trivial or infinite cyclic. (For the reader's convenience, we give a proof of this at the end of the Appendix.) Using this result one easily concludes that, for a group of real-analytic diffeomorphisms of the circle preserving an exceptional minimal set, the intersection of the derived set of every orbit with each connected component of the complement of the Cantor invariant minimal set has at most two points. (In particular, the closure of the union of finitely many orbits cannot be the whole circle.) Indeed, if $]a,b[$ is such a connected component and the orbit of $p \in]a,b[$ intersects $]a,b[$ infinitely many times, then there exists an element $g \in \Gamma$ which fixes $[a,b]$ and such that all the previous intersection points are of the form $g^n(p)$ for some $n \in \mathbf{Z}$. These points form a sequence which converges to the future (respectively to the past) to some point a' (respectively b') in $[a,b]$, and so the intersection of $[a,b]$ with the derived set of the orbit of p is $\{a', b'\}$.

Let us suppose that the orbits by Γ are not dense. Since there is no finite orbit, there must be an exceptional minimal set [13]. The preceding remark then implies that the open set $U = \mathrm{S}^1 \setminus \big(\overline{\Gamma(0)} \cup \overline{\Gamma(\frac{3}{8})}\big)$ is non empty. Moreover, this set is contained in the union of intervals of type $g_n^i(A_1)$ and $g_n^i(B_1)$, where $n \geq 1$ and $i \in \{0, \ldots, 2^n - 1\}$.

Let I be a connected component of U having maximal length. We claim that I is contained in one of the intervals A_1, A_2, B_1 or B_2. Indeed, every connected component of U which is not contained in one of those intervals lies inside $g_n^{-i}(A_2)$ or $g_n^{-i}(B_2)$ for some $n \geq 2$ and $i \in \{1, 3, 5, \ldots, 2^n - 1\}$. Let us consider the first case, the second being analogous. In that case, it is easy to verify that, for all $x \in g_n^{-i}(A_2)$,

$$x, H(x), \ldots, H^{n-2}(x) \notin C_1 \cup C_2, \quad H^{n-1}(x) \in A_2, \quad H^n(x) \in A_1. \tag{18}$$

For each $j \in \{0, \ldots, 2^n - 1\}$ let $H_j^n \colon g_n^{-j}(A_2) \to H^n\big(g_n^{-j}(A_2)\big) \subset]0,1[$ be the injective branch of H^n defined on $g_n^{-j}(A_2)$. If $x \in g_n^{-i}(A_2)$ then $g_n^i(x) = g_1 \circ (H_{2^n-1}^n)^{-1} \circ H_i^n(x)$ (see Figure 3). Thus,

$$(g_n^i)'(x) = \frac{H'\big(H^{n-1}(x)\big)}{H'\big(H^{n-1} \circ (H_{2^n-1}^n)^{-1} \circ H_i^n(x)\big)} \cdot \frac{(H^{n-1})'(x)}{(H^{n-1})'\big((H_{2^n-1}^n)^{-1} \circ H_i^n(x)\big)}.$$

Let us remark that $(H_{2^n-1}^n)^{-1} \circ H_i^n(x)$ belongs to the interval A_1, which is invariant by H. Moreover, since $H^{n-1}(x) \in A_2$ and $n \geq 2$, one has $H^{n-1}\big((H_{2^n-1}^n)^{-1} \circ H_i^n(x)\big) = g_1\big(H^{n-1}(x)\big)$. One then obtains, by (17) and (18),

$$(g_n^i)'(x) \geq \frac{(H^{n-1})'(x)}{\big(\sup_{u \in A_1} H'(u)\big)^{n-1}} > 1.$$

As a consequence we get that the connected component $g_n^i(I) \subset A_2$ of U has greater length than I, which is a contradiction.

We then conclude that the interval I is contained in A_1, A_2, B_1 or B_2. Let us suppose that $I \subset A_1$. Since $f'(x) > 1$ for all $x \in A_1$, the set $f(A_1)$ is

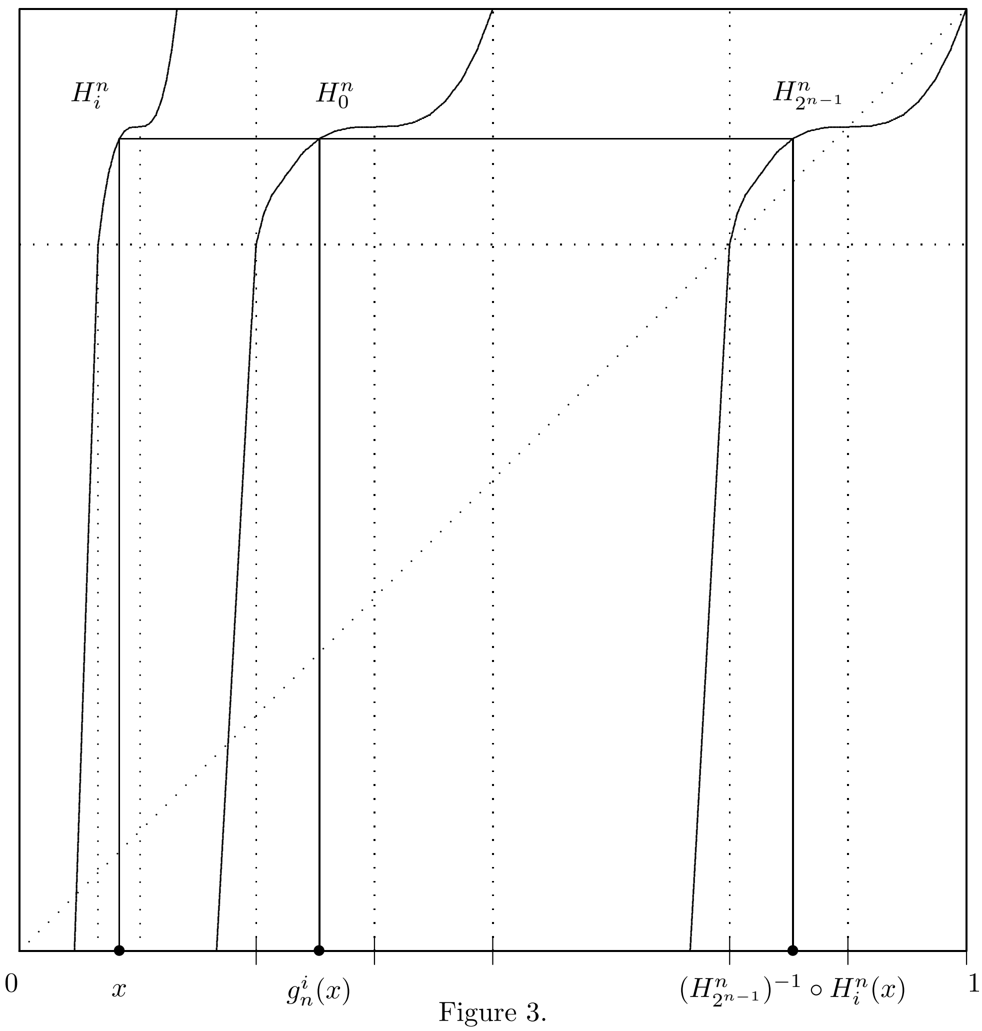

Figure 3.

a connected component U having greater length than I, which is absurd. For $I \subset B_1$, $I \subset A_2$ or $I \subset B_2$, one obtains a contradiction by a similar reasoning using the maps $g_1 f^{-1} g_1$, $g_1 f g_1$ or f^{-1}, respectively. We have then proved that Γ cannot preserve a Cantor minimal set, and since Γ has no finite orbit, this finishes the proof of Claim 1.

From this point of the construction, one can follow N. Kovačević's arguments verbatim. We will reproduce them partially for completeness, but for the interested reader we recommend the reading of the beautiful work [22]. Let us only remark that it is very important to know *a priori* that the orbits by Γ are dense in order to apply these arguments.

Given an oriented interval $]a,b[$ of the circle, let us denote by (a,b) the (non oriented) hyperbolic geodesic joining a and b. We will say that this geodesic is *defined* by the interval $]a,b[$. The open region contained in the unit disk and which is delimited by the geodesics defined by the intervals $g_n^i(A_1)$ and $g_n^i(B_1)$, with $n \in \mathbf{N}$ and $i \in \{0, \ldots, 2^n - 1\}$, will be denoted by Ω. For $g \in \Gamma$ the notation $g(\Omega) = \Omega'$ will be employed when Ω' is the open region delimited by the geodesics defined by the intervals $gg_n(A_1)$ and $gg_n(B_1)$, with $n \in \mathbf{N}$ and $i \in \{0, \ldots, 2^n - 1\}$. If $g(\Omega) = \Omega'$, we will say that Ω' is the *copy* of Ω by g. Let us remark that if Ω_1 and Ω_2 are two copies of Ω and $g \in \Gamma$ is such that $g(\Omega_1) \neq g(\Omega_2)$, then $g(\Omega_1) \cap g(\Omega_2) = \emptyset$. Moreover, it is easy to see that the stabilizer of Ω in Γ coincides with Γ_0.

Claim 2. *If $g \in \Gamma$ and $\sharp \mathrm{Fix}\,(g) \geq 3$, then $g = \mathrm{Id}$.*

Let us suppose that the opposite is true and let $g \in \Gamma$ be a non trivial element which fixes more than two points. Let $I =]a, b[$ be a connected component of $\mathrm{S}^1 \setminus \mathrm{Fix}\,(g)$. We claim that some copy of Ω intersects the geodesic joining a and b. Indeed, in the opposite case, the geodesic (a, b) would be contained inside a copy of Ω, or it would be a component of the boundary of such a copy. In any case, the element g fixes this copy, which is absurd, since the elements of the stabilizer of a copy of Ω are the conjugates of elements of Γ_0, and so they do not have fixed points if they are different from the identity.

Let us then fix a copy Ω_1 of Ω which intersects the geodesic (a, b). We claim that $\Omega \cap (a, b)$ does not contain neither a nor b in its closure. Indeed, in the opposite case the set $g(\Omega_1) \cap \Omega_1$ would be non empty, and so g would be a conjugate of an element of the stabilizer of Ω, which is absurd.

Changing g by g^{-1} if necessary, we may assume that $g^n(x)$ converges to b for all $x \in]a, b[$. Let $c \in]b, a[$ be another fixed point of g (we remark that the fixed points of g are isolated). It is easy to see that for all $x \in]a, c[$ near c, the sequence $\big(g^n(x)\big)$ converges to c. This implies that the sequence of copies of Ω_1 by g^n accumulate on the geodesic joining b to c. However, the argument above shows that there exists a copy Ω_2 of Ω which cuts this last geodesic. We thus have $g^n(\Omega_1) \cap \Omega_2 \neq \emptyset$ for n sufficiently large. However, this implies that g^n is a conjugate of an element of the stabilizer of Ω, which is absurd.

Claim 3. *If $g \in \Gamma$ and $\sharp \mathrm{Fix}\,(g) = 2$, then g is topologically conjugate to a hyperbolic Möbius transformation.*

We must prove that one of the fixed points is (topologically) repelling and the other is (topologically) attracting, and this is an almost direct consequence of the preceding arguments.

Claim 4. *If $g \in \Gamma$ and $\sharp \mathrm{Fix}\,(g) = 1$, then g is topologically conjugate to a parabolic Möbius transformation.*

This is always true, independently of the structure of the group.

Claim 5. *If $g \in \Gamma$ and $\sharp \mathrm{Fix}\,(g) = 0$, then g is topologically conjugate to a finite order rotation.*

For two different copies Ω_1 and Ω_2 of Ω, let us define the *distance* dist (Ω_1, Ω_2) as being equal to 1 plus the minimum number of copies of Ω that a curve must cross in order to go from Ω_1 to Ω_2. Let us also define $\operatorname{dist}(\Omega_1, \Omega_1) = 0$. If $\operatorname{dist}\big(\Omega, g(\Omega)\big) = 0$ then g belongs to the stabilizer of Ω, and so it is a torsion element. If $\operatorname{dist}\big(\Omega, g(\Omega)\big) > 0$ then it is not difficult to see that there exists a copy Ω_1 of Ω such that $\operatorname{dist}\big(\Omega_1, g(\Omega_1)\big)$ is equal to 0 or 1. In the first case, g is a conjugate of an element of the stabilizer of Ω, and so it has finite order. In the second case, it is easy to see that g has order 2.

Claim 6. Γ *is not a Möbius group.*

Indeed, the elements (g_n) generate an Abelian subgroup of Γ which acts freely but is not conjugate to a group of rotations.

Remark 3.2. It would be very interesting in our context to know if every element of Γ is not only topologically but also quasisymmetrically conjugate to a Möbius transformation (see for instance [3]). However, this seems to be a difficult problem.

Remark 3.3. The group we constructed is unfortunately non finitely generated. Let us mention that in [22], N. Kovačević gives examples of finitely generated groups of homeomorphisms of S^1 which are pseudo-Möbius and non Möbius. However, we were not able to give analogous real-analytic examples, and it is indeed not clear that this could be done. Another interesting problem is to find algebraic conditions that, imposed to a pseudo-Möbius group, imply that it is in fact a Möbius group.

To conclude this article, and for completeness, we will give a proof of Hector's unpublished result used in the preceding construction. (Note that a sketch of proof appears already in [12, Proposition 3.9].) The following lemma is well known to the specialists. We give a simple proof "à la Kopell" using control of distortion type arguments [21].

Lemma 3.4. *Let f and g be two real-analytic diffeomorphisms defined on a small neighborhood of the origin of $\mathbf{R}$ and which are tangent to the identity at 0. Let us suppose that for $|x| \leq \varepsilon$ they can be written in the form $f(x) = x + a_i x^i + \cdots$ and $g(x) = x + b_j x^j + \cdots$, and that $f(x) < x$ for all $x > 0$ small enough. If $j > i$, then the sequence $(f^{-n} g f^n)$ converges uniformly to the identity on a non degenerate interval $[0, \varepsilon']$.*

Proof. Let us first remark that the claim of the lemma is quite natural, since from the hypothesis $j > i$ one can see that the Taylor expansion of $f^{-n} g f^n$ about the origin has the form

$$f^{-n} g f^n(x) = x + c_{j,n} x^{j+n} + \cdots .$$

For the proof we first affirm that there exists $\varepsilon'' > 0$ such that $g(x) \geq f(x)$ and $g(x) \leq f^{-1}(x)$ for all $x < \varepsilon''$. Indeed, there exist constants $C_1, C_2 > 0$ such that for all $x < \frac{1}{2}\varepsilon$ one has

$$|g(x) - x| \leq C_1 |x|^j, \quad |f(x) - x| \geq C_2 |x|^i.$$

So, if $\varepsilon'' \leq \max\bigl\{\frac{1}{2}\varepsilon, (C_2/C_1)^{1/(j-i)}\bigr\}$ is small enough, then for all $x < \varepsilon''$ one has

$$g(x) \geq x - C_1|x|^j \geq x - C_2|x|^i \geq f(x).$$

The other inequality $g(x) \leq f^{-1}(x)$ can be obtained in a similar way. In general, an analogous argument shows that for each positive integer N there exists $\varepsilon''(N) > 0$ such that if $y < \varepsilon''(N)$ then

$$|g(y) - y| \leq \frac{1}{N} \max\bigl\{|f(y) - y|, |f^{-1}(y) - y|\bigr\}.$$

For $N \in \mathbf{N}$ let us fix $n(N) \in \mathbf{N}$ such that $f^n(y) \leq \varepsilon''(N)$ for all $y \in [0, \varepsilon']$ and all $n \geq n(N)$. Let us define $\varepsilon' = f(\varepsilon'')$. If $\delta > 0$ is a constant such that $\bigl|\bigl(\log(f')\bigr)'(u)\bigr| \leq \delta$ for all $u \in [0, \varepsilon'']$, then for all $y \in [0, \varepsilon']$, $\bar{y} \in [f(y), f^{-1}(y)]$ and $n \in \mathbf{N}$,

$$\frac{(f^n)'(y)}{(f^n)'(\bar{y})} \leq \exp\biggl(\delta \sum_{k=0}^{n-1} |f^k(y) - f^k(\bar{y})|\biggr) \leq \exp(\varepsilon''\delta).$$

By the mean value theorem, for all $x \in [0, \varepsilon']$ and all $n \geq n(N)$ one has

$$\frac{|f^{-n} g f^n(x) - x|}{|x - f^{\pm 1}(x)|} \leq \exp(\varepsilon''\delta) \frac{|g f^n(x) - f^n(x)|}{|f^n(x) - f^{n \pm 1}(x)|} \leq \frac{\exp(\varepsilon''\delta)}{N},$$

and then one obtains, for some constant $C > 0$,

$$|f^{-n} g f^n(x) - x| \leq \frac{C \exp(\varepsilon''\delta)}{N}.$$

Since this last expression goes to zero as N goes to infinity, the sequence $(f^{-n} g f^n)$ uniformly converges to the identity on $[0, \varepsilon']$. □

Proposition 3.5. *Let* Γ *be a subgroup of* $\mathrm{Diff}_+^{\omega}([0,1])$ *without fixed point on* $]0,1[$*. If* Γ *is neither trivial nor cyclic infinite, then the orbit of every point of* $]0,1[$ *is dense in* $[0,1]$.

Proof. By Szekeres' theorem [37], if Γ is an Abelian group of C^2 diffeomorphisms of $[0,1[$ without fixed points on $]0,1[$, then the restriction of Γ to $]0,1[$ is contained in the flow associated to a C^1 vector field over $[0,1[$. If Γ is Abelian and neither trivial nor cyclic infinite, then the closure of its orbits is the whole interval $[0,1]$, since the corresponding flow is transitive on $]0,1[$. This shows the proposition in the case where Γ is Abelian.

Let us now suppose that Γ is non Abelian. We claim that in this case there exists $\varepsilon' > 0$ and f, g in Γ which do not commute and such that $f^{-n}gf^n$ converges uniformly to the identity on the interval $[0,\varepsilon']$. To prove this we must consider two different cases.

First case. There exists $f \in \Gamma$ such that $f'(0) \neq 1$. Changing f by f^{-1} if necessary, one can suppose that $f'(0) = \lambda < 1$. We claim that there exists a non trivial element $g \in \Gamma$ such that $g'(0) = 1$. Indeed, if this is not the case then every commutator in Γ would be trivial, and so Γ would be Abelian, contradicting our hypothesis. So, let us fix a non trivial element $g \in \Gamma$ such that $g'(0) = 1$. It is easy to see that f and g do not commute. By Koenig's linearization theorem ([4, p. 31]), modulo conjugacy of f and g by a real-analytic diffeomorphism, we may assume that $f(x) = \lambda x$ for all $0 \leq x \leq \varepsilon'$. So, after conjugacy, for $x \in [0,\varepsilon']$ one has the uniform convergence

$$\lim_{n\to+\infty} f^{-n}gf^n(x) = \lim_{n\to+\infty} \frac{g(\lambda^n x)}{\lambda^n} = g'(0)x = x.$$

Second case. Every element $f \in \Gamma$ is tangent to the identity at 0. Let f be an element of Γ whose contact order i with respect to the identity is minimal. We claim that there exists an element $g \in \Gamma$ having contact order with respect to the identity greater than i. Indeed, if all the non trivial elements of Γ have the same contact order with the identity at 0, then the corresponding contact order of all commutators in Γ would be greater than i (due to the hypothesis of tangency to the identity), and so Γ would be Abelian, contradicting our hypothesis. So, by Lemma 3.4, $f^{-n}gf^n$ converges uniformly to the identity on an interval $[0,\varepsilon']$.

To finish the proof of the proposition, we claim that for all $x \in]0,1[$ the point 0 is contained in the interior of the closure K of the orbit by x. Indeed, by the hypothesis of non existence of fixed point on $]0,1[$, the origin belongs to K. Let us suppose that 0 does not belong to the interior of K, and let us fix a connected component I of the complement of K contained in $[0,\varepsilon']$. Since K is invariant by Γ, the preceding claim implies that $f^{-n}gf^n(I) = I$ for n large enough. So, one has $g\big(f^n(I)\big) = f^n(I)$ for infinitely many positive integer numbers n, which contradicts the analyticity of g.

Let us now consider the infimum $u_0 \in [0,1]$ of the set of points $u \in [0,1]$ such that $[0,u] \subset \mathrm{K}$. The preceding claim gives $u_0 > 0$. Since u_0 is a fixed point by Γ, this implies that $u_0 = 1$, that is the orbit of x by Γ is dense. Since $x \in]0,1[$ is arbitrary, this finishes the proof. □

We can finally give a proof for Hector's result.

Theorem (Hector). *If* Γ *is a subgroup of* $\mathrm{Diff}_+^\omega(\mathrm{S}^1)$ *having an exceptional minimal set, then for all point* $a \in \mathrm{S}^1$ *the stabilizer* Γ_a *of* a *in* Γ *is trivial or cyclic infinite.*

Proof. The group Γ_a can be viewed as a group of real-analytic diffeomorphisms of the interval $[a, a+1]$. If Γ_a is neither trivial nor cyclic infinite then, by analyticity and the preceding proposition, the orbits by Γ_a of all but a finite number of points $p \in]a, a+1[$ are dense near p. As a consequence, the orbits by Γ of all but a finite number of points $b \neq a$ in S^1 are dense on some open set of S^1. However, this contradicts the hypothesis of existence of an invariant exceptional minimal set. □

An interesting corollary of this last theorem is the fact that every subgroup of $\mathrm{Diff}_+^\omega(\mathrm{S}^1)$ having a Cantor minimal set is countable, which is not necessarily true for subgroups of $\mathrm{Diff}_+^\infty(\mathrm{S}^1)$ having such a minimal set.

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