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LIPSCHITZ SPACES AND HARMONIC MAPPINGS

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Abstract. In [11] the author proved that every quasiconformal harmonic mapping between two Jordan domains with $C^{1,\alpha}$, $0<\alpha\le 1$, boundary is bi-Lipschitz, providing that the domain is convex. In this paper we avoid the restriction of convexity. More precisely we prove: any quasiconformal harmonic mapping between two Jordan domains Ω_j, $j=1,2$, with $C^{j,\alpha}$, $j=1,2$ boundary is bi-Lipschitz.

1. Introduction and notation

A function w is called *harmonic* in a region D if it has the form $w=u+iv$ where u and v are real-valued harmonic functions in D. If D is simply-connected, then there are two analytic functions g and h defined on D such that w has the representation

$$w=g+\overline{h}.$$

If w is a harmonic univalent function, then by Lewy's theorem (see [17]), w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism. If k is an analytic function and w is a harmonic function, then $w\circ k$ is harmonic. However, $k\circ w$ is not harmonic in general.

Let

$$P(r,x-\varphi)=\frac{1-r^2}{2\pi(1-2r\cos(x-\varphi)+r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function w defined on the unit disc $\mathbf{U}:=\{z:|z|<1\}$ has the following representation

$$w(z)=P[w_b](z)=\int_0^{2\pi}P(r,x-\varphi)w_b(e^{ix})\,dx, \tag{1.1}$$

where $z=re^{i\varphi}$ and w_b is a bounded integrable function defined on the unit circle $S^1:=\{z:|z|=1\}$.

Let $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$. We will consider two matrix norms

$$|A|=\max\{|Az|:z\in\mathbf{R}^2,\ |z|=1\}\ \text{ and }\ |A|_2=\Big(\sum_{i,j}a_{i,j}^2\Big)^{1/2},$$

and the matrix function

$$l(A)=\min\{|Az|:|z|=1\}.$$

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Let $w = u+iv\colon D \mapsto G$, $D, G \subset \mathbf{C}$, be differentiable at $z \in D$. By $\nabla w(z)$ we denote the matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. For the matrix ∇w we have

$$|\nabla w| = |w_z| + |w_{\bar z}|, \quad |\nabla w|_2 = (|w_x|^2 + |w_y|^2)^{1/2} = \sqrt{2}(|w_z|^2 + |w_{\bar z}|^2)^{1/2}$$

and

$$l(\nabla w) = ||w_z| - |w_{\bar z}||.$$

Thus

$$|\nabla w| \le |\nabla w|_2 \le \sqrt{2}|\nabla w|. \tag{1.2}$$

A homeomorphism $w\colon D \mapsto G$, where D and G are subdomains of the complex plane $\mathbf{C}$, is said to be K-quasiconformal (K-q.c.), $K \ge 1$, if w is absolutely continuous on a.e. horizontal and a.e. vertical line and

$$\left|\frac{\partial w}{\partial x}\right|^2 + \left|\frac{\partial w}{\partial y}\right|^2 \le 2KJ_w \quad \text{a.e. on } D, \tag{1.3}$$

where J_w is the Jacobian of w (cf. [1], pp. 23–24). Notice that condition (1.3) can be written as

$$|w_{\bar z}| \le k|w_z| \quad \text{a.e. on } D \text{ where } k = \frac{K-1}{K+1}, \text{ i.e., } K = \frac{1+k}{1-k}\ ,$$

or in its equivalent form

$$\frac{(|\nabla w|)^2}{K} \le J_w \le K(l(\nabla w))^2. \tag{1.4}$$

We will focus on harmonic quasiconformal mappings between Jordan domains with smooth boundary and will investigate their Lipschitz character.

Recall that a mapping $w\colon D \mapsto G$ is said to be C-Lipschitz ($C > 1$) (c-co-Lipschitz) ($0 < c$) if

$$|w(z_2) - w(z_1)| \le C|z_2 - z_1|,\ z_1, z_2 \in D,$$
$$(c|z_2 - z_1| \le |w(z_2) - w(z_1)|,\ z_1, z_2 \in D).$$

2. Background and statement of the main result

It is well known that a conformal mapping of the unit disk onto itself has the form

$$w = e^{i\varphi}\frac{z-a}{1-z\bar a}, \quad \varphi \in [0, 2\pi),\ |a| < 1.$$

By the Riemann mapping theorem there exists a Riemann conformal mapping of the unit disk onto a Jordan domain $\Omega = \operatorname{int}\gamma$. By Caratheodory's theorem it has a continuous extension to the boundary. Moreover, if $\gamma \in C^{n,\alpha}$, then the Riemann conformal mapping has $C^{n,\alpha}$ extension to the boundary, see [29]. Conformal mappings are quasiconformal and harmonic. Hence quasiconformal harmonic mappings are a natural generalization of conformal mappings. The first characterization of quasiconformal harmonic mappings was established by Martio in [18]. Hengartner and Schober have shown that for a given second dilatation ($a = \overline{f_{\bar z}}/f_z$ with $\|a\| < 1$) there exist a q.c. harmonic mapping f between two Jordan domains with analytic boundary [4, Theorem 4.1]. Recently there has been a number of authors who are working on the topic. Using the result of Heinz [5]: If w is a harmonic diffeomorphism of the unit disk onto itself with $w(0) = 0$, then

$$|w_z|^2 + |w_{\bar{z}}|^2 \geq \frac{1}{\pi^2}.$$

Martio [18] observed that every quasiconformal harmonic mapping of the unit disk onto itself is co-Lipschitz. Mateljević, Pavlović and Kalaj have shown that the family of quasiconformal and harmonic mappings share with conformal mappings the following property: if w is harmonic q.c. mapping of the unit disk onto a Jordan domain with rectifiable boundary, then w has absolutely continuous extension to the boundary, see [7]. What happens if the boundary of a co-domain is "smoother than rectifiable"? Pavlović [23] proved that every quasiconformal selfmapping of the unit disk is Lipschitz continuous, using the Mori's theorem on the theory of quasiconformal mappings. Partyka and Sakan [22] yielded explicit Lipschitz and co-Lipschitz constants depending on a constant of quasiconformality. Since the composition of a harmonic mapping and of a conformal mapping is itself harmonic, using Kellogg's theorem (Proposition 3.3) these theorems have generalizations to the class of mappings from arbitrary Jordan domain with $C^{1,\alpha}$ boundary to the unit disk. However, the composition of a conformal and a harmonic mapping is not, in general, a harmonic mapping. This means, in particular, that the results of this kind for arbitrary co-domain do not follow from the case of the unit disk and Kellogg's theorem. The situation of co-domain different from the unit disk has been firstly considered in [15], and it has been shown that every harmonic quasiconformal mapping of the half-plane onto itself is bi-Lipschitz. Moreover, in [15] it has been given two characterizations of those mappings, the first one in terms of a boundary mapping, using the Hilbert transforms of the derivative of the boundary function, and the second one deals with an integral representation, with the help of analytic functions. The facts concerning Hilbert transforms can be found in [30]. Concerning those situations (the disk and the half-plane) see also [16]. The author [8] extended Heinz theorem [5] for the harmonic mappings from the unit disk onto a convex domain. This in turn implies that quasiconformal harmonic mappings of the unit disk onto a convex domain are co-Lipschitz [9]. Using a new method the results [23] have been extended properly by the author and Mateljević in [11], [19], and [14]. The extensions are the following:

Let Ω_1 and Ω be Jordan domains, let $\mu \in (0, 1]$, $a \in \Omega_1$, and let $f\colon \Omega_1 \mapsto \Omega$ be a harmonic homeomorphism.

(a) If f is K-q.c. and $\partial\Omega_1, \partial\Omega \in C^{1,\mu}$, then f is Lipschitz with Lipschitz constant $c_0(\Omega_1, \Omega, K, a, w(a))$. Moreover, for almost every $t \in \partial\Omega_1$ there exists

$$\lim_{z \xrightarrow{\angle} t, z\in\Omega_1} \nabla f(z) = \nabla f(t). \tag{2.1}$$

(b) If f is q.c. and if $\partial\Omega_1, \partial\Omega \in C^{1,\mu}$ and Ω is convex, then f is bi-Lipschitz.

(c) If Ω_1 is the unit disk, Ω is convex, and $\partial\Omega \in C^{1,\mu}$, then f is quasiconformal if and only if its boundary function f_b is bi-Lipschitz and the Hilbert transformation of its derivative is in L^∞.

(d) If f is q.c. and if Ω is convex, then the boundary functions f_b is bi-Lipschitz in the Euclidean metric and the Cauchy transform $C[f_b']$ of its derivative is in L^∞.

(e) If f is q.c. and if Ω is convex, then the inverse of the boundary function g_b is Lipschitz in the Euclidean metric and the Cauchy transform $C[g_b']$ of its derivative is in L^∞.

Concerning the items (a), (b) and (c) we refer to [11], and for the items (d) and (e), see [19] and [21].

Let now f be a quasiconformal C^2 diffeomorphism from a $C^{1,\alpha}$ Jordan domain Ω_1 onto a $C^{2,\alpha}$ Jordan domain Ω.

(f) If there exists a constant M such that

$$|\Delta f| \leq M|f_z \cdot f_{\bar{z}}|, \quad z \in \Omega, \tag{2.2}$$

then f has bounded partial derivatives. In particular, f is a Lipschitz mapping. For the item (f) we refer to [14].

The result (f) has been generalized in [13] as follows:

(g) If there exist constants M and N such that

$$|\Delta f| \leq M|\nabla f|^2 + N, \quad z \in \Omega, \tag{2.3}$$

then f has bounded partial derivatives in Ω_1. In particular, f is a Lipschitz mapping in Ω_1.

For several dimensional generalizations we refer to [10], [20], [2] and [12].

Because of the lack of a generalization of the Heinz theorem for non-convex domains, it was intrigue to investigate the q.c. harmonic mappings of the unit disk onto an image domain that is not convex. Namely it has been an open problem until now that, if the assumption of convexity on an image domain Ω was important or not in proving the theorem that a harmonic q.c. mapping of the unit disk onto Ω is bi-Lipschitz.

In the following theorem we avoid the restriction of convexity.

Theorem 2.1. (The main theorem) *Let $w = f(z)$ be a K-quasiconformal harmonic mapping between a Jordan domain Ω_1 with $C^{1,\alpha}$ boundary and a Jordan domain Ω with $C^{2,\alpha}$ boundary. Let, in addition, $a \in \Omega_1$ and $b = f(a)$. Then w is bi-Lipschitz. Moreover, there exists a positive constant $c = c(K, \Omega_1, \Omega, a, b) \geq 1$ such that*

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1. \tag{2.4}$$

3. Proof of the main theorem

A key of the proof is Lemma 3.2, which could be considered as a global version of the following well-known lemma:

Lemma 3.1. (Hopf's boundary point lemma [26], [6]) *Let u satisfy $\Delta u \geq 0$ in D and $u \leq M$ in D, $u(P) = M$ for some $P \in \partial D$. Assume that P lies on the boundary of a ball $B \subset D$. If u is continuous on $D \cup P$ and if the outward directional derivative $\frac{\partial u}{\partial n}$ exists at P, then $u \equiv M$ or*

$$\frac{\partial u}{\partial n} > 0.$$

Lemma 3.2. *Let u satisfy $\Delta u \geq 0$ in $R_\varrho = \{z : \varrho \leq |z| < 1\}$, $0 < \varrho < 1$, u be continuous on $\overline{R_\varrho}$, $u < 0$ in R_ϱ, $u(t) = 0$ for $t \in S^1$. Assume that the radial* *derivative* $\frac{\partial u}{\partial r}$ *exists for almost every* $t \in S^1$*. Let* $M(u, \varrho) := \max_{|z|=\varrho} u(z)$*. Then for the positive constant*

$$c(u, \varrho) = \frac{2M(u, \varrho)}{\varrho^2(1 - e^{1/\varrho^2 - 1})} \tag{3.1}$$

it holds

$$\frac{\partial u(t)}{\partial r} > c(u, \varrho) \ \text{ for a.e. } t \in S^1. \tag{3.2}$$

Proof. Consider the auxiliary function $h_\varrho^A(z) = e^{-A|z|^2} - e^{-A}$, where $A > 0$ is a constant to be chosen later. Then

$$\Delta h_\varrho^A(z) = 4Ae^{A|z|^2}(A|z|^2 - 1).$$

Hence it has the property that $h_\varrho^A(z) > 0$, $z \in R_\varrho$, and that

$$\Delta h_\varrho^A \geq 0, \varrho \leq |z| \leq 1, \tag{3.3}$$

if

$$A \geq \varrho^{-2}, \ \text{ for example, } A = \varrho^{-2}. \tag{3.4}$$

The function $h_\varrho^A(z)$ is of class C^2 in R_ϱ, and

$$h_\varrho^A(z) = 0 \ \text{ on } \ S^1. \tag{3.5}$$

The function $v_\varrho^A = u + \varepsilon h_\varrho^A(z)$, $\varepsilon > 0$, is of class C^2 in the interior of R_ϱ and continuous in R_ϱ. Moreover, by (3.5),

$$v_\varrho^A \leq 0 \ \text{ on } \ S^1. \tag{3.6}$$

As $M(u, \varrho) < 0$, we can choose a constant ε so that

$$M(u, \varrho) + \varepsilon(e^{-A\varrho^2} - e^{-A}) \leq 0.$$

For example,

$$\varepsilon = \frac{M(u, \varrho)}{e^{-A} - e^{-A\varrho^2}}. \tag{3.7}$$

Then we have

$$v_\varrho^A \leq 0 \ \text{ also on } \ S(0, \varrho). \tag{3.8}$$

By the hypothesis, $\Delta u \geq 0$ in R_ϱ, and by (3.3) it follows

$$\Delta v_\varrho^A > 0, \ \ z \in R_\varrho. \tag{3.9}$$

Now (3.6), (3.8), and (3.9) imply that $v_\varrho^A \leq 0$ holds in the whole of R_ϱ. This follows from the elementary fact that v_ϱ^A cannot have a positive maximum in the interior of R_ϱ. But $v_\varrho^A \leq 0$ in R_ϱ and $v_\varrho^A = 0$ at $t \in S^1$ imply that

$$0 \leq \lim_{R \to 1-0} \frac{v_\varrho^A(Rt) - v_\varrho^A(t)}{R - 1} = \frac{\partial v_\varrho^A(t)}{\partial r} = \frac{\partial u(t)}{\partial r} + \varepsilon \frac{\partial h_\varrho^A(t)}{\partial r}.$$

Furthermore,

$$\min_{s \in S^1} \frac{\partial h_\varrho^A(s)}{\partial r} = -2Ae^{-A} < 0.$$

Thus for almost every $t \in S^1$ it holds

$$\frac{\partial u(t)}{\partial r} \geq -\varepsilon \min_{s\in S^1} \frac{\partial h_\varrho^A(s)}{\partial r} = \frac{2AM(u,\varrho)}{1-e^{(1-\varrho^2)A}} =: c(u,\varrho) > 0. \tag{3.10}$$

□

To continue we need the following propositions:

Proposition 3.3. (Kellogg [3]) *If a domain $D = \mathrm{Int}(\Gamma)$ is $C^{1,\alpha}$ and ω is a conformal mapping of* **U** *onto D, then ω' and $\ln \omega'$ are in Lip_α. In particular, $|\omega'|$ is bounded from above and below on* **U** *by two positive constants.*

Let Γ be a smooth Jordan curve and $\beta(s)$ the angle of the tangent as a function of arc length. We say that Γ has a Dini-continuous curvature if $\beta'(s)$ is continuous and

$$|\beta'(s_2) - \beta'(s_1)| \leq \omega_1(s_2 - s_1)\ (s_1 < s_2),$$

where $\omega_1(x)$ is an increasing function that satisfies

$$\int_0^1 \frac{\omega_1(s)}{s}\, ds < \infty.$$

The next proposition is due to Kellogg and to Warschawski.

Proposition 3.4. [24, Theorem 3.6] *Let ω be a conformal mapping of the unit disk onto a Jordan domain that is bounded by a Jordan curve with Dini-continuous curvature. Then $\omega''(z)$ has a continuous extension to $\overline{\mathbf{U}}$. In particular, $|\omega''|$ is bounded from above on* **U**.

Notice that if Γ is $C^{2,\alpha}$, then Γ has Dini-continuous curvature. We will finish the proof of Theorem 2.1 using the following lemma.

Lemma 3.5. *Let $w = f(z)$ be a K-quasiconformal harmonic mapping of the unit disk onto a $C^{2,\alpha}$ Jordan domain Ω such that $w(0) = a \in \Omega$. Then there exists a constant $C(K,\Omega,a) > 0$ such that*

$$|\frac{\partial w}{\partial r}(t)| \geq C(K,\Omega,a) \text{ for almost every } t \in S^1.$$

Proof. Let g be a conformal mapping of Ω onto the unit disk with $g(a) = 0$. Take $w_1 = g \circ w$. Then

$$\Delta w_1 = 4g''(w) w_z \cdot w_{\bar z} + g'(w)\Delta w = 4g''(w) w_z \cdot w_{\bar z} = 4\frac{g''}{|g'|^2} w_{1z} \cdot w_{1\bar z}. \tag{3.11}$$

Combining (3.11) and (1.4) we obtain

$$|\Delta w_1| \leq \frac{|g''|}{|g'|^2}|(|\nabla w_1|^2 - l(\nabla w_1)^2) \leq \left(1 - \frac{1}{K^2}\right)\frac{|g''|}{|g'|^2}|\nabla w_1|^2. \tag{3.12}$$

Let $h(z) = |w_1|^2$. Let us find two constants $B > 0$ and $\varrho \in (0,1)$ such that the function

$$\varphi(z) := \chi(h(z)) = \frac{1}{B}(e^{Bh(z)} - e^B)$$

is subharmonic on $\{z : \varrho < |z| < 1\}$. Clearly $\varphi(z) \leq 0$. On the other hand, we have

$$\Delta\varphi = \chi''(h)|\nabla h|^2 + \chi'(h)\Delta h. \tag{3.13}$$

Furthermore,

$$\Delta h = 2|\nabla w_1|_2^2 + 2\langle \Delta w_1, w_1\rangle. \tag{3.14}$$

Let $w_1 = \rho s$, $\rho = |w_1|$, $s = e^{i\psi}$. Then

$$|\nabla h| = 2\rho|\nabla\rho|. \tag{3.15}$$

To continue, observe that

$$\nabla w_1 = (\nabla\rho)^t s + \rho\nabla s,$$

and thus

$$|\nabla w_1\, l|^2 = |\rho\nabla s\, l|^2 + |\nabla\rho\, l\cdot s|^2 + 2\rho\nabla\rho l\, \langle\nabla s\, l, s\rangle\,,\ l\in\mathbf{R}^2.$$

Hence

$$|\nabla w_1\, l|^2 = \rho^2|\nabla s\, l|^2 + |\nabla\rho\, l|^2. \tag{3.16}$$

Choose l_1, $|l_1| = 1$ so that $\nabla s l_1 = 0$. Then by (3.16) we infer

$$|\nabla w_1\, l_1| \le |\nabla\rho\, l_1|.$$

According to the definition of quasiconformal mappings we obtain

$$K^{-1}|\nabla w_1| \le |\nabla\rho|. \tag{3.17}$$

From (3.15) and (3.17) it follows that

$$|\nabla h| \ge \frac{2\rho}{K}|\nabla w_1|. \tag{3.18}$$

Combining (1.2), (3.12), (3.13), (3.14) and (3.18) we obtain

$$\Delta\varphi \ge \left(\chi''\frac{4\rho^2}{K^2} + 2\chi' - 2\left(1 - \frac{1}{K^2}\right)\chi'\frac{|g''|}{|g'|^2}\right)|\nabla w_1|^2. \tag{3.19}$$

Furthermore,

$$\chi'(h) = e^{Bh} \tag{3.20}$$

and

$$\chi''(h) = Be^{Bh}. \tag{3.21}$$

By (3.19), (3.20) and (3.21) we obtain

$$\Delta\varphi \ge \left(B\frac{4\rho^2}{K^2} + 2 - 2\left(1 - \frac{1}{K^2}\right)\frac{|g''|}{|g'|^2}\right)e^{Bh(z)}|\nabla w_1|^2. \tag{3.22}$$

As $w_1 = \rho s$ is K-quasiconformal self-mapping of the unit disk with $w_1(0) = 0$, by Mori's theorem ([27]) it satisfies the double inequality

$$\left|\frac{z}{4^{1-1/K}}\right|^K \le \rho \le 4^{1-1/K}|z|^{1/K}. \tag{3.23}$$

By (3.23) for $\varrho \le |z| \le 1$ where

$$\varrho := 4^{-K}, \tag{3.24}$$

we have

$$\rho \ge 4^{1-K^2-K}. \tag{3.25}$$

Now we choose B such that

$$\frac{4B\rho^2}{K^2} + 2 - 2\left(1 - \frac{1}{K^2}\right)\frac{|g''|}{|g'|^2} \ge 0,$$

i.e., in view of Propositions 3.3 and 3.4, and (3.25), take, for example,

$$B := \max\left\{\frac{1}{2}\sup_{z\in\Omega}\left|1-\left(1-\frac{1}{K^2}\right)\frac{|g''|}{|g'|^2}\right|K^2 4^{K^2+K-1}, 1\right\}. \tag{3.26}$$

According to Lemma 3.2 and to (2.1), the function

$$\varphi(z) = \chi(h(z)) = \frac{1}{B}(e^{Bh(z)} - e^B)$$

satisfies

$$\frac{\partial\varphi}{\partial R}(t) = e^{Bh(t)}\left\langle g'(w(t))\cdot\frac{\partial w}{\partial R}(t), w_1(t)\right\rangle \geq c(\varphi,\varrho)$$

almost everywhere in S^1, where $c(\varphi,\varrho)$ is defined by (3.1). On the other hand, by the right hand side inequality in (3.23) it follows that

$$\varphi(z) \leq \frac{1}{B}(e^{4^{-\frac{2}{K}}B} - e^B) \text{ for } |z| = \varrho. \tag{3.27}$$

Thus

$$M(\varphi,\varrho) = \max_{|z|=\varrho}\varphi(z) \leq \frac{1}{B}(e^{4^{-\frac{2}{K}}B} - e^B) < 0. \tag{3.28}$$

According to (3.1) and (3.2) it follows that

$$|\frac{\partial w}{\partial r}(t)| \geq \frac{e^{-B}c(\varphi,\varrho)}{\max\{|g'(\zeta)| : \zeta\in\partial\Omega\}} = \frac{2e^{-B}M(\varphi,\varrho)}{\varrho^2(1-e^{1/\varrho^2-1})|g'|_\infty} > 0$$

almost everywhere in S^1. By (3.24), (3.26) and (3.28), we can take

$$C(K,\Omega,a) = \frac{2e^{-B}M(\varphi,\varrho)}{\varrho^2(1-e^{1/\varrho^2-1})|g'|_\infty},$$

where $C(K,\Omega,a)$ do not depend on $w = f(z)$. □

Proof of Theorem 2.1. In view of item a) from the background of this paper, it is enough to prove that w is co-Lipschitz continuous (under the above conditions). Moreover, by Proposition 3.3 the unit disk could be taken as the domain of the mapping.

We will consider two cases:

1. Case: $w \in C^1(\overline{\mathbf{U}})$. Let $l(\nabla w)(t) = ||w_z(t)| - |w_{\bar z}(t)||$. As w is K-q.c., according to Lemma 3.5 we have

$$l(\nabla w)(t) \geq \frac{|\nabla w(t)|}{K} \geq \frac{|\frac{\partial w}{\partial r}(t)|}{K} \geq \frac{C(K,\Omega,a)}{K} \tag{3.29}$$

for $t \in S^1$.

Since w is a harmonic diffeomorphism, by the Lewy theorem [17] ($|w_z| > 0$), it defines a bounded subharmonic function

$$S(z) := \left|\frac{\overline{w_{\bar z}}}{w_z}\right| + \left|\frac{1}{w_z}\cdot\frac{C(K,\Omega,a)}{K}\right| \tag{3.30}$$

on the unit disk. According to (3.29), $S(z)$ is bounded on the unit circle by 1. By the maximum principle, this implies that S is bounded by 1 on the whole unit disk.

This in turn implies that for every $z \in \mathbf{U}$,

$$l(\nabla w)(z) \geq \frac{C(K,\Omega,a)}{K}. \tag{3.31}$$

2. Case: $w \notin C^1(\overline{\mathbf{U}})$.

Definition 3.6. Let G be a domain in $\mathbf{C}$ and let $a \in \partial G$. We will say that $G_a \subset G$ is a neighborhood of a if there exists a disk $D(a,r) := \{z : |z-a| < r\}$ such that $D(a,r) \cap G \subset G_a$.

Let $t = e^{i\beta} \in S^1$, then $w(t) \in \partial\Omega$. Let γ be an arc-length parametrization of $\partial\Omega$ with $\gamma(s) = w(t)$. Since $\partial\Omega \in C^{2,\alpha}$, there exists a neighborhood Ω_t of $w(t)$ with $C^{2,\alpha}$ Jordan boundary such that

$$\Omega_t^\tau := \Omega_t + i\gamma'(s)\cdot\tau \subset \Omega, \text{ and } \partial\Omega_t^\tau \subset \Omega \text{ for } 0 < \tau \le \tau_t \ (\tau_t > 0). \tag{3.32}$$

An example of a family Ω_t^τ such that $\partial\Omega_t^\tau \in C^{1,\alpha}$ and with the property (3.32) has been given in [11]. An easy modification yields a family of Jordan domains Ω_t^τ with $\partial\Omega_t^\tau \in C^{2,\alpha}$, $0 \le \tau \le \tau_t$ with the property (3.32).

Let $a_t \in \Omega_t$ be arbitrary. Then $a_t + i\gamma'(s)\cdot\tau \in \Omega_t^\tau$. Take $U_\tau = f^{-1}(\Omega_t^\tau)$. Let η_t^τ be a conformal mapping of the unit disk onto U_τ such that $\eta_t^\tau(0) = f^{-1}(a_t + i\gamma'(s)\cdot\tau)$ and $\arg \frac{d\eta_t^\tau}{dz}(0) = 0$. Then the mapping

$$f_t^\tau(z) := f(\eta_t^\tau(z)) - i\gamma'(s)\cdot\tau$$

is harmonic K-quasiconformal mapping of the unit disk onto Ω_t satisfying the condition $f_t^\tau(0) = a_t$. Moreover,

$$f_t^\tau \in C^1(\overline{\mathbf{U}}).$$

Using the first case $w \in C^1(\overline{\mathbf{U}})$, it follows that

$$|\nabla f_t^\tau(z)| \ge C(K, \Omega_t, a_t).$$

On the other hand,

$$\lim_{\tau\to 0+} \nabla f_t^\tau(z) = \nabla(f\circ\eta_t)(z)$$

on the compact sets of $\mathbf{U}$ as well as

$$\lim_{\tau\to 0+} \frac{d\eta_t^\tau}{dz}(z) = \frac{d\eta_t}{dz}(z),$$

where η_t is a conformal mapping of the unit disk onto $U_0 = f^{-1}(\Omega_t)$ with $\eta_t(0) = f^{-1}(a_t)$. It follows that

$$|\nabla f_t(z)| \ge C(K, \Omega_t, a_t).$$

Using the Schwartz's reflexion principle to the mapping η_t, and using the formula

$$\nabla(f\circ\eta_t)(z) = \nabla f \cdot \frac{d\eta_t}{dz}(z),$$

it follows that in some neighborhood $\tilde{U}_t$ of $t \in S^1$ ($D(t, r_t) \cap \mathbf{U} \subset \tilde{U}_t$ for some $r_t > 0$), the function f satisfies the inequality

$$|\nabla f(z)| \ge \frac{C(K, \Omega_t, a_t)}{\min\{|\eta_t(\zeta)| : \zeta \in \partial\tilde{U}_t \cap S^1\}} =: \tilde{C}(K, \Omega_t, a_t) > 0. \tag{3.33}$$

Since S^1 is a compact set, it can be covered by a finite family $\partial\tilde{U}_{t_j} \cap S^1 \cap D(t, r_t/2)$, $j = 1, \dots, m$. It follows that the inequality

$$|\nabla f(z)| \ge \min\{\tilde{C}(K, \Omega_{t_j}, a_{t_j}) : j = 1, \dots, m\} =: \tilde{C}(K, \Omega, a) > 0 \tag{3.34}$$

holds in the annulus

$$\tilde{R} = \left\{ z : 1 - \frac{\sqrt{3}}{2} \min_{1 \le j \le m} r_{t_j} < |z| < 1 \right\} \subset \bigcup_{j=1}^{m} \tilde{U}_{t_j}.$$

This implies that the subharmonic function S defined in (3.30) is bounded in $\mathbf{U}$. According to the maximum principle it is bounded by 1 in the whole unit disk. This in turn implies again (3.31) and consequently

$$\frac{C(K,\Omega,a)}{K}|z_1 - z_2| \le |w(z_1) - w(z_2)|, \quad z_1, z_2 \in \mathbf{U}. \tag{3.35}$$ □

Corollary 3.7. *If w is a q.c. harmonic mapping of the unit disk onto a $C^{2,\alpha}$ Jordan domain Ω, then* $\operatorname{ess\,sup}\{J_w(z), z \in \mathbf{U}\} > 0$.

Example 3.8. A mapping $w = P[e^{i(x+\sin x)}](z)$, $z \in \mathbf{U}$, is a harmonic diffeomorphism of the unit disk onto itself having smooth extension to the boundary and

$$\begin{aligned} 0 \le J_w(-1) &\le |\frac{\partial w}{\partial r}(re^{i\varphi})|_{r=1,\varphi=\pi}| \cdot |\frac{\partial w}{\partial \varphi}(e^{i\varphi})|_{\varphi=\pi} \\ &= |\frac{\partial w}{\partial r}(re^{i\varphi})|_{r=1,\varphi=\pi} \cdot |(1 + \cos\varphi|_{\varphi=\pi})| = 0, \end{aligned}$$

i.e., $J_w(-1) = 0$. Hence the condition of quasiconformality in Corollary 3.7 is essential.

Remarks 3.9. It seems natural that the assumption $\partial\Omega \in C^{2,\alpha}$ in the main theorem can be replaced by $\partial\Omega \in C^{1,\alpha}$, however, we do not have a proof of this fact. It remains an open problem, whether the norm of the first derivative of a harmonic diffeomorphism between the unit disk and a smooth Jordan domain Ω is bounded below by a constant depending on Ω. The result of this kind was proved by Heinz [5] for the case of Ω being the unit disk, and by the author in [8] for Ω being a convex domain. In this paper it was proved that the result holds for harmonic quasiconformal mappings without the restriction on convexity of the co-domain.

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