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NON-DEGENERACY OF PERTURBED SOLUTIONS OF SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The equation $-\Delta u + F(V(\varepsilon x), u) = 0$ is considered in $\mathbf{R}^n$. For small $\varepsilon > 0$ it is shown to possess, under appropriate conditions, a non-degenerate solution u_ε in $H^2(\mathbf{R}^n)$. It is shown that the linearised operator T_ε at the solution satisfies $\|T_\varepsilon^{-1}\| = O(\varepsilon^{-2})$ as $\varepsilon \to 0$.

1. Introduction

In this paper we consider the question of non-degeneracy of certain solutions of a partial differential equation. A solution is non-degenerate when the linearised problem at the solution, in appropriate function spaces, defines an invertible linear operator. Throughout this article, by an invertible operator, with specified Banach spaces as domain and codomain, we shall mean a linear surjective homeomorphism.

The notion of non-degeneracy depends on the choice of spaces. In this paper we shall be concerned with problems that can be posed using the Sobolev spaces $W^{2,2}(\mathbf{R}^n)$ and $L^2(\mathbf{R}^n)$ as domain and codomain respectively. For conciseness we denote these spaces by H^2 and L^2 respectively. The same problems might be posable in spaces of classically differentiable functions and the question of non-degeneracy in this setting can also arise. There are connections between the two notions of non-degeneracy which do not seem to have been fully explored. In this paper we shall study non-degeneracy for Sobolev spaces only.

It seems safe to assume that the applications of non-degeneracy, once established, are many. We could mention the stable behaviour of non-degenerate solutions under perturbations guaranteed by the implicit function theorem. Moreover for equations of the type we shall consider, non-degenerate solutions give rise to multibump solutions, see for example [1] (using spaces of differentiable functions), or [3] (using Sobolev spaces).

Let us first consider the problem

$$-\Delta u + F(u) = 0 \tag{1.1}$$

in $\mathbf{R}^n$. Under conditions on F to be specified the non-linear operator $\Gamma(u) = -\Delta u + F(u)$ is well-defined from H^2 to L^2 and will have a well-defined Fréchet derivative, the linear operator $v \mapsto D\Gamma(u)v = -\Delta v + \frac{dF}{du}(u)v$.

Let us assume that we have a solution ϕ of (1.1). It is highly implausible for ϕ to be a non-degenerate solution as the partial derivatives $D_k\phi$ will be in the kernel of

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$D\Gamma(\phi)$ if they lie in H^2. Similarly, since the problem (1.1) commutes with rotations, we expect the functions $\nabla\phi(x)\cdot Tx$ to be in the kernel whenever T is a skew-symmetric matrix. However these functions will be 0 if ϕ is spherically symmetric.

We shall say that a spherically symmetric solution ϕ is quasi-non-degenerate if the partial derivatives $D_j\phi(x)$ belong to H^2, they are linearly independent, span the kernel of $D\Gamma(\phi)$, and the range of $D\Gamma(\phi)$ is the orthogonal complement in L^2 to its kernel.

Quasi-non-degenerate solutions are easy to construct in one dimension. We consider $-u'' + F(u) = 0$ where F is a smooth function such that $F(0) = 0$, $F'(0) > 0$ and $\Phi(u) = -\int F(u)\,du$ satisfies $\sup_{u>0}\Phi(u) > \Phi(0)$. Then the solution $\phi(x)$ is quasi-non-degenerate where $x \mapsto (\phi(x), \phi'(x))$ is the phase-plane trajectory in the region $u > 0$ which tends to the saddle point $(0,0)$ as $x \to \pm\infty$.

In higher dimensions a range of quasi-non-degenerate solutions is known for the equation

$$-\Delta u + u - u^p = 0 \tag{1.2}$$

More precisely it is known that the ground state solution, defined to be the solution with minimum energy $\int(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1})\,dx$, exists and is quasi-non-degenerate for all integers $p > 1$ if $n = 1, 2$ and for $1 < p < (n+2)/(n-2)$ if $n > 2$. See the papers [2], [5] and [7].

The possibility arises of obtaining non-degenerate solutions by perturbing (1.1), when a quasi-non-degenerate solution is known, to a problem explicitly containing x. Various perturbation schemes have been studied, for example that of [3, Sections 4, 5], which generates non-degenerate solutions to a more general equation of the type $-\Delta u + F(x, u, \nabla u) = 0$. Another paper [4] considered perturbations of a different nature. Suppose we have a one-parameter continuum of problems

$$-\Delta u + F(a, u) = 0 \tag{1.3}$$

where a belongs to the real interval I, and suppose for each value $a \in I$ we have a quasi-non-degenerate solution $\phi_a(x)$. An example showing how such a continuum can be constructed is given in Section 4. Now we perturb (1.3) to

$$-\Delta u + F(V(\varepsilon x), u) = 0 \tag{1.4}$$

where $V(x)$ is a function with range in I and $\varepsilon > 0$. The difficulty of this scheme arises from the weak nature of the convergence to a problem of the form (1.3) as $\varepsilon \to 0$.

In the previous paper [4, Section 3] it was shown how to obtain a solution u_ε in H^2 of (1.4) for all sufficiently small ε, say, for $0 < \varepsilon < \varepsilon_0$, and its asymptotic form was described. This is a so-called single-bump solution and it is asymptotic to $\phi_a(x - \frac{b}{\varepsilon})$ (for a certain b and $a = V(b)$) as $\varepsilon \to 0$. These results are summarised in Section 2, Theorem 2.6. The question of non-degeneracy of the solution u_ε was not studied in [4]. The main object of this paper is provide a clear proof of non-degeneracy for the single-bump solutions of (1.4) together with an estimate of the blow-up of the inverse of the derivative as $\varepsilon \to 0$. In fact we shall show that the linearised operator

$$T_\varepsilon := -\Delta + \frac{\partial F}{\partial u}(V(\varepsilon x), u_\varepsilon)\colon H^2 \to L^2$$

is invertible for sufficiently small $\varepsilon > 0$. Moreover, its inverse satisfies an estimate

$$\|T_\varepsilon^{-1}\| = O\left(\frac{1}{\varepsilon^2}\right)$$

as ε tends to 0.

The precise version of this result, Theorem 3.1, is stated and proved in Section 3. The proof consists of two distinct and independent steps:

1. It is shown that T_ε is a Fredholm operator of index 0.
2. It is shown that for every sequence ε_ν with limit 0 and for every bounded sequence $v_\nu \in H^2$ such that $\varepsilon_\nu^{-2} T_{\varepsilon_\nu} v_\nu \to 0$ in L^2, a subsequence of v_ν tends to 0 in H^2.

The first step shows that T_ε, if injective, is also surjective. The second, and by far lengthier step, shows that T_ε is injective and its inverse, considered as an operator defined on its range, satisfies the claimed estimate.

The proof of the second step considers a decomposition $v_\nu(\cdot - t_\nu) = \sigma_\nu \cdot \nabla\phi_a + w_\nu$ where $\sigma_\nu \in \mathbf{R}^n$, ϕ_a is the ground state solution occurring in the asymptotic form of u_ε, the function w_ν is orthogonal to the partial derivatives of ϕ_a, and t_ν a certain translation vector. It is then shown that a subsequence can be found for which $\sigma_\nu \to 0$ and $w_\nu \to 0$ in H^2. Hilbert space methods, for example weak compactness of the unit ball, and self-adjointness of the operator T_ε and related operators, weigh heavily in the proof.

2. Principal assumptions

In this section we state clearly the conditions to be imposed on F, V and ϕ_a. We summarise material from [4] that will be needed. We also define precisely the solution u_ε referred to in the introduction.

Properties of F. We assume that F is a C^2 map satisfying the following growth conditions:

$$\begin{aligned} |F(a,u)|, \left|\frac{\partial F}{\partial a}(a,u)\right|, \left|\frac{\partial^2 F}{\partial a^2}(a,u)\right| &\le C\left(|u| + |u|^{\alpha_1}\right), \\ \left|\frac{\partial F}{\partial u}(a,u)\right|, \left|\frac{\partial^2 F}{\partial u \partial a}(a,u)\right| &\le C\left(1 + |u|^{\alpha_2}\right), \\ \left|\frac{\partial^2 F}{\partial u^2}(a,u)\right| &\le C\left(1 + |u|^{\alpha_3}\right), \end{aligned}$$

where C is chosen uniformly for a in a bounded interval and the exponents α_i are non-negative (in addition $\alpha_1 \ge 1$). No upper limits are placed on α_i if $n \le 4$ whereas for $n \ge 5$ we assume that

$$\alpha_1 \le \frac{n}{n-4}, \quad \alpha_2 \le \frac{4}{n-4}, \quad \alpha_3 < \frac{8-n}{n-4}.$$

Under these growth conditions F, $\dfrac{\partial F}{\partial a}$, $\dfrac{\partial^2 F}{\partial a^2}$, $\dfrac{\partial F}{\partial u}$, $\dfrac{\partial^2 F}{\partial u \partial a}$ and $\dfrac{\partial^2 F}{\partial u^2}$ define Nemitskii operators

$$\boldsymbol{F},\ \boldsymbol{F}_a,\ \boldsymbol{F}_{aa}\colon L^\infty \times H^2 \to L^2,$$
$$\boldsymbol{F}_u,\ \boldsymbol{F}_{ua}\colon L^\infty \times H^2 \to \mathscr{L}(H^2, L^2),$$
$$\boldsymbol{F}_{uu}\colon L^\infty \times H^2 \to \mathscr{L}_2(H^2 \times H^2, L^2)$$

by means of

$$\boldsymbol{F}(m,u) = F(m,u)\,,\ \boldsymbol{F}_u(m,u)v = \frac{\partial F}{\partial u}(m,u)v$$

and so on. In these formulas we use $\mathscr{L}_k$ to denote the appropriate space of symmetric k-linear mappings.

Under these conditions, the Nemitskii operators induced by F and its derivatives have the following boundedness property (see [4]):

Lemma 2.1. *The maps $\boldsymbol{F}$, $\boldsymbol{F}_a$, $\boldsymbol{F}_{aa}$, $\boldsymbol{F}_u$, $\boldsymbol{F}_{ua}$ and $\boldsymbol{F}_{uu}$ map bounded subsets of $L^\infty \times H^2$ to bounded subsets of the appropriate function or operator space.*

We shall often consider sequences of functions indexed by ν ("n" is reserved for the dimension of $\mathbf{R}^n$). It is understood that ν is an integer and limits, where they occur, are for $\nu \to \infty$.

The following convergence properties were proved in [4] and will be used repeatedly later on.

Lemma 2.2. *Let $m_\nu \in L^\infty$ be a bounded sequence that tends pointwise to $m \in L^\infty$. Let u_ν in H^2 converge to $u \in H^2$ and let $v, w \in H^2$. Then*

$$\boldsymbol{F}(m_\nu, u_\nu) \to \boldsymbol{F}(m,u),\ \boldsymbol{F}_a(m_\nu, u_\nu) \to \boldsymbol{F}_a(m,u),\ \boldsymbol{F}_u(m_\nu, u_\nu)v \to \boldsymbol{F}_u(m,u)v,$$
$$\boldsymbol{F}_{uu}(m_\nu, u_\nu)(v,w) \to \boldsymbol{F}_{uu}(m,u)(v,w).$$

Lemma 2.3. *Let $m_\nu \in L^\infty$ be a bounded sequence that tends pointwise to $m \in L^\infty$ and let $u_\nu \in H^2$ converge weakly to $u \in H^2$. Then, for any bounded sequence $v_\nu \in H^2$,*

$$\boldsymbol{F}_u(m_\nu, u_\nu)v_\nu - \boldsymbol{F}_u(m,u)v_\nu \longrightarrow 0$$

in the weak topology on the dual of H^2.

Lemma 2.4. *Let m_ν be a bounded family in L^∞, and u_ν, v_ν and w_ν be bounded sequences in H^2 such that either*

1. *$u_\nu - v_\nu$ is convergent in H^2 and w_ν converges weakly to 0, or*
2. *$u_\nu - v_\nu$ converges weakly to 0 in H^2 and w_ν is convergent.*

Then

$$\big(\boldsymbol{F}_u(m_\nu, u_\nu) - \boldsymbol{F}_u(m_\nu, v_\nu)\big)w_\nu \to 0$$

in L^2. Furthermore, $\boldsymbol{F}_u(m,u) - \boldsymbol{F}_u(m,v)$ is a compact operator for each $m \in L^\infty$, and u, $v \in H^2$.

Properties of ϕ_a. The function $\phi_a(x)$ is a solution to $-\Delta u + F(a,u) = 0$ in H^2 and has the following properties:

1. $\phi_a(x) = \Phi_a(r)$ is spherically symmetric.
2. $\displaystyle\int \frac{\partial F}{\partial a}(a, \Phi_a(r))\, \Phi_a'(r) r\, dx \neq 0.$
3. ϕ_a and its first derivatives have exponential decay.

4. ϕ_a is a quasi-non-degenerate solution, that is, the operator

$$-\Delta + \frac{\partial F}{\partial u}(a, \phi_a(x))\colon H^2 \to L^2$$

has as its kernel the space spanned by the n partial derivatives $D_j\phi_a(x)$, which are assumed to be independent, and its range is the space in L^2 orthogonal to its kernel. This implicitly says that the partial derivatives belong to H^2.

These properties hold in the model case of the non-linear Schrödinger equation (1.2) described in the introduction, see [2], [5] and [7].

Properties of V. The function V is C^2 with its range in the interval I. It and its first partial derivatives are bounded, while its second partial derivatives have polynomial growth.

Positivity assumption. There exists $\delta > 0$ such that

$$\frac{\partial F}{\partial u}(a, 0) > \delta$$

for all a in the range of V.

For later reference we shall need a version of Wang's Lemma (see [6, 3]):

Lemma 2.5. *Let f_ν be a family of measurable functions such that*

$$0 < \delta < f_\nu(x) < K$$

for all ν and constants δ and K. Let μ_ν be a sequence of non-negative numbers and let v_ν be a sequence in H^2 such that

$$-\Delta v_\nu + (f_\nu(x) + \mu_\nu)v_\nu \to 0$$

in L^2. Then $v_\nu \to 0$ in H^2.

Under these conditions the following theorem, proved in [4], holds.

Theorem 2.6. *Let b be a non-degenerate critical point of V and let $a = V(b)$. Then, for sufficiently small $\varepsilon > 0$, the equation $-\Delta u + F(V(\varepsilon x), u) = 0$ has a solution of the form*

$$u_\varepsilon(x) = \phi_a\left(x - \frac{b}{\varepsilon} + s_\varepsilon\right) + \varepsilon^2 w_\varepsilon\left(x - \frac{b}{\varepsilon} + s_\varepsilon\right)$$

where $s_\varepsilon \in \mathbf{R}^n$, $w_\varepsilon \in H^2$ and w_ε is orthogonal in L^2 to the partial derivatives $D_j\phi_a$. Both s_ε and w_ε depend continuously on ε. As ε tends to 0, s_ε tends to 0 and w_ε tends to a computable function $\eta \in H^2$, which is the unique solution $v = \eta(x)$ orthogonal to the partial derivatives $D_j\phi_a$ of the problem

$$-\Delta v + \frac{\partial F}{\partial u}(a, \phi_a(x))v = -\frac{1}{2}\frac{\partial F}{\partial a}(a, \phi_a(x))(H(b)x \cdot x) \tag{2.1}$$

where $H(b)$ is the Hessian matrix of V at the point b. Finally the solution u_ε is the unique one possessing these asymptotic properties.

3. Non-degeneracy of the solutions

Now that we have defined u_ε we can state precisely the main conclusion of this paper.

Theorem 3.1. *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the solution u_ε obtained under the conditions of Theorem 2.6 is non-degenerate, that is, the operator*

$$T_\varepsilon := -\Delta + \frac{\partial F}{\partial u}(V(\varepsilon x), u_\varepsilon)$$

from H^2 to L^2 is invertible. Moreover, we have the following bound on its inverse:

$$\|T_\varepsilon^{-1}\| \leq C\left(\frac{1}{\varepsilon^2}\right)$$

where C is independent of ε.

Proof. We refer the reader to the outline given in Section 1. The proof is in two rather unequal steps. The first step is to show that T_ε is a Fredholm operator of index 0.

Let

$$A_\varepsilon := -\Delta + \frac{\partial F}{\partial u}(V(\varepsilon x), 0) \colon H^2 \to L^2$$

By the positivity assumption A_ε is a self-adjoint operator with domain H^2 satisfying $A_\varepsilon > -\Delta + \delta$, and hence an invertible operator from H^2 to L^2. Now T_ε is a compact perturbation of A_ε, since $T_\varepsilon - A_\varepsilon$ is given by multiplication by

$$f(x) := \frac{\partial F}{\partial u}(V(\varepsilon x), u_\varepsilon) - \frac{\partial F}{\partial u}(V(\varepsilon x), 0)$$

and therefore defines a compact operator from H^2 to L^2 by Lemma 2.4. Hence T_ε is a Fredholm operator of index 0. It follows that if T_ε is injective it is also surjective.

We turn to the second step as outlined in Section 1. We wish to show that for every sequence ε_ν with limit 0 and for every bounded sequence $v_\nu \in H^2$ such that $\varepsilon_\nu^{-2} T_{\varepsilon_\nu} v_\nu \to 0$ in L^2 a subsequence of v_ν tends to 0 in H^2. That this implies the required estimate for the inverse T_ε^{-1} is a consequence of a simple functional analytic lemma.

Lemma 3.2. *Let $(\Gamma_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be a family of bounded linear operators from a Banach space E to a Banach space F. Assume it satisfies the following condition:*

(C) *If ε_ν is a sequence in the interval $]0, \varepsilon_0[$ with limit 0 and x_ν a bounded sequence in E such that $\lim \Gamma_{\varepsilon_\nu} x_\nu = 0$ then x_ν has a subsequence with limit 0.*

There then exists $\varepsilon_1 > 0$, such that for all $0 < \varepsilon < \varepsilon_1$ the operator Γ_ε is injective and its inverse as an operator from its range R_ε to E satisfies $\|\Gamma_\varepsilon^{-1}\|_{\mathscr{L}(R_\varepsilon, E)} \leq K$ where the constant K is independent of ε.

Proof of Lemma 3.2. Clearly there exists ε_1 such that Γ_ε is injective for $0 < \varepsilon < \varepsilon_1$. If it did not exist there would exist a sequence $\varepsilon_\nu \to 0$ and a sequence of *unit vectors* x_ν such that $\Gamma_{\varepsilon_\nu} x_\nu = 0$, contradicting (C).

Suppose the bound on the inverse does not hold for any ε_1 and K. Then we can find a sequence $\varepsilon_\nu \to 0$ and a sequence of *unit vectors* $y_\nu = \Gamma_{\varepsilon_\nu} x_\nu$ in the range R_{ε_ν} such that $\|x_\nu\| \to \infty$. Let $v_\nu = x_\nu / \|x_\nu\|$. Then $\Gamma_{\varepsilon_\nu} v_\nu = y_\nu / \|x_\nu\| \to 0$, contradicting (C). This ends the proof of Lemma 3.2. □

We continue with step 2 of the proof of Theorem 3.1. Let $\varepsilon_\nu \to 0$, let v_ν be a bounded sequence in H^2 and assume that

$$\varepsilon_\nu^{-2}\left(-\Delta + \frac{\partial F}{\partial u}(V(\varepsilon_\nu x), u_{\varepsilon_\nu})\right) v_\nu \to 0$$

in L^2. We recall the asymptotic form for u_ε (Theorem 2.6) and apply it to u_{ε_ν}. For notational convenience, we abbreviate s_{ε_ν} to s_ν and w_{ε_ν} to w_ν. We now replace x by $x + \frac{b}{\varepsilon_\nu} - s_\nu$ and find

$$\varepsilon_\nu^{-2}\left(-\Delta + \frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a + \varepsilon_\nu^2 w_\nu\Big)\right) v_\nu\left(x + \frac{b}{\varepsilon_\nu} - s_\nu\right) \to 0$$

in L^2.

We recall that our objective is to show that a subsequence of v_ν tends to 0 in H^2. In the sequel we shall repeatedly select a subsequence and always denote it by v_ν (and similarly for ε_ν etc.) using a phrase such as "going to a subsequence we may assume".

Let W be the subspace of H^2 orthogonal in the L^2 sense to the n partial derivatives $D_j\phi_a$. We write

$$v_\nu\left(\cdot + \frac{b}{\varepsilon_\nu} - s_\nu\right) = \sigma_\nu \cdot \nabla\phi_a + \gamma_\nu h_\nu$$

where $\sigma_\nu \in \mathbf{R}^n$, $h_\nu \in W$, $\|h_\nu\|_{H^2} = 1$ and $\gamma_\nu \geq 0$. Step 2 of the proof will be completed by showing that, after going to a subsequence, σ_ν and γ_ν both have the limit 0.

We begin by observing that since v_ν is bounded in H^2 the sequences γ_ν and σ_ν are bounded. We may therefore assume (going to a subsequence) that $\gamma_\nu \to \gamma_0$, $\sigma_\nu \to \sigma_0$ and, exploiting weak compactness of the unit ball in Hilbert space, that $h_\nu \to h_0$ weakly in H^2.

In view of the equation $(-\Delta + \frac{\partial F}{\partial u}(a,\phi_a))\nabla\phi_a = 0$ we now have

$$\begin{aligned}
&\varepsilon_\nu^{-2}\gamma_\nu\left(-\Delta + \frac{\partial F}{\partial u}\Big(V(\varepsilon(x-s_\nu)+b), \phi_a + \varepsilon_\nu^2 w_\nu\Big)\right) h_\nu \\
&+ \varepsilon_\nu^{-2}\left(\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a + \varepsilon_\nu^2 w_\nu\Big) - \frac{\partial F}{\partial u}\Big(a,\phi_a\Big)\right)(\nabla\phi_a \cdot \sigma_\nu) \to 0
\end{aligned} \tag{3.1}$$

in L^2. It is our objective to show that the left-hand side of (3.1) has a computable limit which gives rise to a limit equation. The computation will occupy the bulk of step 2 of the proof of Theorem 3.1 and concludes with the limit equation (3.6).

We first claim that the second term in (3.1) tends in L^2 to

$$\frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta\,\nabla\phi_a\cdot\sigma_0 + \frac{1}{2}\frac{\partial^2 F}{\partial u\partial a}(a,\phi_a)(H(b)x\cdot x)\nabla\phi_a\cdot\sigma_0$$

where $H(b)$ is the Hessian matrix of V and η is the solution to (2.1) (see Theorem 2.6). To see this we expand it into

$$\begin{aligned}
&\varepsilon_\nu^{-2}\left(\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a+\varepsilon_\nu^2 w_\nu\Big) - \frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a\Big)\right)(\nabla\phi_a\cdot\sigma_\nu) \\
&+ \varepsilon_\nu^{-2}\left(\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a\Big) - \frac{\partial F}{\partial u}\Big(a,\phi_a\Big)\right)(\nabla\phi_a\cdot\sigma_\nu)
\end{aligned} \tag{3.2}$$

The first summand of (3.2) can be written as the integral

$$\int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a + \tau\varepsilon_\nu^2 w_\nu\Big) w_\nu \nabla\phi_a\cdot\sigma_\nu\, d\tau \tag{3.3}$$

For fixed $\tau \in [0,1]$

$$\frac{\partial^2 F}{\partial u^2}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a + \tau\varepsilon_\nu^2 w_\nu\Big)\eta\nabla\phi_a\cdot\sigma_0 \longrightarrow \frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta\nabla\phi_a\cdot\sigma_0$$

according to Lemma 2.2. Moreover,

$$\frac{\partial^2 F}{\partial u^2}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a + \tau\varepsilon_\nu^2 w_\nu\Big)w_\nu(\nabla\phi_a\cdot\sigma_\nu) - \frac{\partial^2 F}{\partial u^2}\Big(V(\varepsilon_\nu(x-s_\nu)+b), \phi_a + \tau\varepsilon_\nu^2 w_\nu\Big)\eta(\nabla\phi_a\cdot\sigma_0)$$

tends to 0 owing to Lemma 2.1, the boundedness of V, and the limits $w_\nu \to \eta$ in H^2 and $\nabla\phi_a\cdot\sigma_\nu \to \nabla\phi_a\cdot\sigma_0$ in H^2. Therefore, the integrand in (3.3) tends to

$$\frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta\nabla\phi_a\cdot\sigma_0$$

in L^2 at fixed τ. Also, the L^2-norm of the integrand stays bounded independently of τ and ν (again by Lemma 2.1), so the dominated convergence theorem for L^2-valued integrals shows that the integral tends in L^2 to

$$\frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta\nabla\phi_a\cdot\sigma_0.$$

The second summand of (3.2) can be written as the integral

$$\varepsilon_\nu^{-1}\int_0^1 \frac{\partial^2 F}{\partial u\partial a}\Big(V(\tau\varepsilon_\nu(x-s_\nu)+b,\phi_a\Big)\big(\nabla\phi_a\cdot\sigma_\nu\big)\nabla V(\tau\varepsilon_\nu(x-s_\nu)+b)\cdot(x-s_\nu)\,d\tau$$

Since, by assumption, $\nabla V(b)=0$, this is equal to

$$\int_0^1\!\!\int_0^1 \frac{\partial^2 F}{\partial u\partial a}\Big(V(\tau\varepsilon_\nu(x-s_\nu)+b,\phi_a\Big)\big(\nabla\phi_a\cdot\sigma_\nu\big)H(\rho\tau\varepsilon_\nu(x-s_\nu)+b)(x-s_\nu)\cdot(x-s_\nu)\tau\,d\rho\,d\tau$$

Now, $H(x)$ has polynomial growth, $\frac{\partial^2 F}{\partial u\partial a}(V(\tau\varepsilon_\nu(x-s_\nu)+b,\phi_a)$ is bounded uniformly with respect to τ and ν as x goes to infinity, because of our growth conditions, and $(\nabla\phi_a\cdot\sigma_\nu)$ has uniform exponential decay as x goes to infinity (since σ_ν is a bounded sequence). Hence, for fixed τ, the integrand converges to

$$\frac{\partial^2 F}{\partial u\partial a}(a,\phi_a)\big(\nabla\phi_a\cdot\sigma_0\big)(H(b)x\cdot x)\tau$$

in L^2, and is also bounded uniformly with respect to τ and ν by a fixed function in L^2. We can therefore apply the dominated convergence theorem for L^2-valued integrals and the double integral tends to

$$\frac{1}{2}\frac{\partial^2 F}{\partial u\partial a}(a,\phi_a)\big(\nabla\phi_a\cdot\sigma_0\big)(H(b)x\cdot x).$$

This proves our claim that the second term of (3.1) has a limit in L^2.

Next we consider the first term of (3.1). We claim that the multiplier $\varepsilon_\nu^{-2}\gamma_\nu$ is bounded. Suppose, for the sake of contradiction, that $\varepsilon_\nu^{-2}\gamma_\nu$ is unbounded. Going to a subsequence we may assume that $\varepsilon_\nu^2\gamma_\nu^{-1}\to 0$ and multiplying (3.1) by it and knowing that the second term of (3.1) converges in L^2, we obtain

$$\left(-\Delta + \frac{\partial F}{\partial u}(V(\varepsilon_\nu(x-s_\nu)+b),\phi_a+\varepsilon_\nu^2 w_\nu)\right)h_\nu \to 0 \tag{3.4}$$

in L^2. Now we have

$$\frac{\partial F}{\partial u}(a,\phi_a)h_\nu \to \frac{\partial F}{\partial u}(a,\phi_a)h_0$$

weakly in L^2 since multiplication by $\frac{\partial F}{\partial u}(a,\phi_a)$ is a norm-continuous linear operator from H^2 to L^2. Moreover

$$\left(\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b),\phi_a+\varepsilon_\nu^2 w_\nu\Big)-\frac{\partial F}{\partial u}(a,\phi_a)\right)h_\nu \to 0$$

in the weak topology of the dual of H^2 by Lemma 2.3. Finally $\Delta h_\nu \to \Delta h_0$ in the sense of distributions. We conclude from (3.4) that

$$-\Delta h_0 + \frac{\partial F}{\partial u}(a,\phi_a)h_0 = 0.$$

But $h_0 \in W$ so that $h_0 = 0$ by property 4 of ϕ_a. We deduce by Lemma 2.4 that

$$\left(\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b),\phi_a+\varepsilon_\nu^2 w_\nu\Big)-\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b),0\Big)\right)h_\nu \to 0$$

in L^2, so that now (3.4) yields

$$\left(-\Delta+\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b),0\Big)\right)h_\nu \to 0$$

in L^2. Now the positivity assumption and Wang's Lemma give $h_\nu \to 0$ in H^2 thus contradicting the assumption that $\|h_\nu\|_{H^2} = 1$. This contradiction implies that $\varepsilon_\nu^{-2}\gamma_\nu$ is bounded as claimed. One consequence of this is that $\gamma_\nu \to 0$; thus one of the objectives of step 2 is attained.

We may now assume, going once more to a subsequence, that $\varepsilon_\nu^{-2}\gamma_\nu \to c \geq 0$. Armed with this knowledge we return to (3.1) reminding ourselves at this point that now we do not necessarily have $h_0 = 0$. We do, however, still have that

$$\left(-\Delta+\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x-s_\nu)+b),\phi_a+\varepsilon_\nu^2 w_\nu\Big)\right)h_\nu \to \left(-\Delta+\frac{\partial F}{\partial u}(a,\phi_a)\right)h_0$$

in the sense of distributions. Passing to the limit in (3.1) we find that

$$\begin{aligned}&\left(-\Delta+\frac{\partial F}{\partial u}(a,\phi_a)\right)ch_0+\frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta(\nabla\phi_a\cdot\sigma_0)\\ &+\frac{1}{2}\frac{\partial^2 F}{\partial u\partial a}(a,\phi_a)(H(b)x\cdot x)\nabla\phi_a\cdot\sigma_0 = 0.\end{aligned} \tag{3.5}$$

Now recall that η satisfies the equation

$$-\Delta\eta+\frac{\partial F}{\partial u}(a,\phi_a)\eta = -\frac{1}{2}\frac{\partial F}{\partial a}(a,\phi_a)(H(b)x\cdot x).$$

Differentiating this with respect to x gives the vector-valued equation

$$\begin{aligned}&-\Delta(\nabla\eta)+\frac{\partial F}{\partial u}(a,\phi_a)\nabla\eta+\frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta\nabla\phi_a\\ &= -\frac{1}{2}\frac{\partial^2 F}{\partial a\partial u}(a,\phi_a)(H(b)x\cdot x)\nabla\phi_a-\frac{\partial F}{\partial a}(a,\phi_a)H(b)x.\end{aligned}$$

Taking the inner product with σ_0 and using (3.5) gives the final form of the limit equation that arises from (3.1):

$$\left(-\Delta + \frac{\partial F}{\partial u}(a, \phi_a)\right)(ch_0 + \nabla\eta \cdot \sigma_0) = -\frac{\partial F}{\partial a}(a, \phi_a)H(b)x \cdot \sigma_0. \tag{3.6}$$

The next objective is to show that $\sigma_0 = 0$. By (3.6) the function $\frac{\partial F}{\partial a}(a, \phi_a)H(b)x \cdot \sigma_0$ is in the range of $-\Delta + \frac{\partial F}{\partial u}(a, \phi_a)$, which is the space W. It follows that σ_0 satisfies the linear equation system

$$\int \left(\frac{\partial F}{\partial a}(a, \phi_a)H(b)x \cdot \sigma_0\right) D_j\phi_a(x)\, dx = 0, \quad j = 1, \dots, n.$$

We claim that this implies $\sigma_0 = 0$. Write $\sigma_0 = (\sigma_1, \dots, \sigma_n)$. The inner product $H(b)x \cdot \sigma_0$ is given by

$$H(b)x \cdot \sigma_0 = \sum_{i,k} \sigma_i \left(x_k H_{k,i}\right)$$

where the $H_{k,i}$ are the coefficients of the matrix $H(b)$. Moreover, ϕ_a is spherically symmetric, $\phi_a(x) = \Phi_a(r)$, so our system can be written as

$$\int \left(\sum_{i,k} \sigma_i \left(x_k H_{k,i}\right)\right) \frac{\partial F}{\partial a}(a, \Phi_a(r)) \frac{\Phi_a'(r) x_j}{r}\, dx = 0, \quad j = 1, \dots, n.$$

Spherical symmetry causes terms involving mixed products $x_j x_k$, $j \neq k$, to vanish, leading to the simpler equation

$$\left(\sum_i \sigma_i H_{j,i}\right) \int \frac{\partial F}{\partial a}(a, \Phi_a(r))\Phi_a'(r) \frac{x_j^2}{r}\, dx = 0$$

for all j. It is easily seen that the integral is independent of j, and so its value is

$$C := \frac{1}{n} \int \frac{\partial F}{\partial a}(a, \Phi_a(r))\Phi_a'(r) r\, dx$$

which is non-zero by assumption. Therefore, our system reduces to

$$C \sum_i \sigma_i H_{j,i} = 0$$

for all j, and since $H(b)$ is an invertible matrix, this implies $\sigma_0 = 0$, hence our claim.

We have now arrived at a subsequence v_ν for which $\sigma_\nu \to 0$ and $\varepsilon_\nu^{-2}\gamma_\nu$ is bounded. The latter implies $\gamma_\nu \to 0$. Since $v_\nu\big(\,\cdot + \frac{b}{\varepsilon_\nu} - s_\nu\big) = \nabla\phi_a \cdot \sigma_\nu + \gamma_\nu h_\nu$ and $\|h_\nu\|_{H^2} = 1$ we conclude that $v_\nu \to 0$ in H^2. This concludes step 2 of the proof and therefore the whole proof is complete. □

We remark that for $c < 2$ it is not the case that $\|T_\varepsilon^{-1}\| = O(\varepsilon^{-c})$. Indeed if $c < 2$ then $\lim_{\varepsilon \to 0} \varepsilon^{-c} T_\varepsilon v_\varepsilon = 0$ in L^2, where $v_\varepsilon(x) = D_j\phi_a(x - \frac{b}{\varepsilon} + s_\varepsilon)$.

4. Remark on the assumptions

The non-vanishing of the integral

$$I := \int \frac{\partial F}{\partial a}(a, \Phi_a(r))\Phi_a'(r)\, r\, dx \tag{4.1}$$

was used in the proof of Theorem 3.1 (as well as in the proof of the very existence of u_ε). This condition may seem rather technical and meaningless. We can however derive it from a rather natural assumption in which the continuum of ground states ϕ_a plays an interesting role. We shall assume that the ϕ_a form a smooth continuum with respect to a, an assumption that we did not need before, since we were always working at fixed a. This is often the case since, in practice, the functions ϕ_a are usually obtained by scaling from the case $a = 1$.

Proposition 4.1. *In addition to all previous hypotheses on ϕ_a, assume that it is a C^1 function of a. Then the integral I is non-zero if and only if*

$$\frac{d}{da}\left(\int |\nabla\phi_a|^2\,dx\right) \neq 0.$$

Proof. We shall in fact establish the identity

$$I := \int \frac{\partial F}{\partial a}(a,\Phi_a(r))\Phi_a'(r)\,r\,dx = -\frac{d}{da}\left(\int |\nabla\phi_a|^2\,dx\right). \tag{4.2}$$

By spherical symmetry we have

$$\int \frac{\partial F}{\partial a}(a,\Phi_a(r))\Phi_a'(r)\,r\,dx = n\int \frac{\partial F}{\partial a}(a,\phi_a(x))x_jD_j\phi_a(x)\,dx$$

for each j. Differentiating the equation $-\Delta\phi_a + F(a,\phi_a) = 0$ with respect to a gives

$$-\Delta\left(\frac{\partial\phi_a}{\partial a}\right) + \frac{\partial F}{\partial u}(a,\phi_a)\frac{\partial\phi_a}{\partial a} + \frac{\partial F}{\partial a}(a,\phi_a) = 0$$

and therefore

$$\begin{aligned} I &= n\int\left(\Delta - \frac{\partial F}{\partial u}(a,\phi_a)\right)\left(\frac{\partial\phi_a}{\partial a}\right)x_jD_j\phi_a(x)\,dx \\ &= n\int\frac{\partial\phi_a}{\partial a}\left(\Delta - \frac{\partial F}{\partial u}(a,\phi_a)\right)(x_jD_j\phi_a)\,dx \end{aligned}$$

using self-adjointness. We expand

$$\Delta(x_jD_j\phi_a) = x_j\Delta D_j\phi_a + 2D_j^2\phi_a$$

and since the partial derivatives $D_j\phi_a$ belong to the kernel of $-\Delta + \dfrac{\partial F}{\partial u}(a,\phi_a)$, we are left with

$$I = 2n\int\frac{\partial\phi_a}{\partial a}D_j^2\phi_a\,dx.$$

This is true for each j, so

$$I = 2\int\frac{\partial\phi_a}{\partial a}\Delta\phi_a\,dx = -2\int\nabla\phi_a\cdot\nabla\left(\frac{\partial\phi_a}{\partial a}\right)dx = -\frac{d}{da}\left(\int|\nabla\phi_a|^2\,dx\right),$$

and the proof is complete. □

We now give more details about how a continuum can be obtained from a single solution for a fixed value of a, and how the result of the previous proposition applies then.

Assume we are given a non-trivial, spherically symmetric solution $\psi(x)$ in H^2 of an equation

$$-\Delta u + G(u) = 0$$

where G satisfies our regularity and growth conditions. For a in a bounded interval I, we put

$$\phi_a(x) = a^\mu \psi(a^\nu x)$$

for positive exponents μ and ν. This is obviously smooth with respect to a. Moreover,

$$\Delta\phi_a = a^{\mu+2\nu}\Delta\psi(a^\nu x) = a^{\mu+2\nu}G(a^{-\mu}\phi_a).$$

Therefore ϕ_a solves $-\Delta u + F(a,u) = 0$ where

$$F(a,u) = a^{\mu+2\nu}G(a^{-\mu}u).$$

We see that $F(1,u) = G(u)$. The function F satisfies our growth conditions. According to Proposition 4.1, the non-vanishing of the integral 4.1 reduces to

$$0 \neq \frac{d}{da}\left(a^{2\mu+2\nu}\int |\nabla\psi(a^\nu x)|^2\,dx\right) = (2\mu+2\nu-n\nu)a^{2\mu+2\nu-n\nu-1}\int |\nabla\psi|^2\,dx.$$

So a necessary and sufficient condition for the non-vanishing of this integral is simply

$$2\mu + 2\nu - n\nu \neq 0.$$

Under this simple assumption, we can apply Theorems 2.6 and 3.1, using a potential V with range in I.

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