Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 35, 2010, 379–387

LIPSCHITZ-TYPE SPACES AND HARMONIC MAPPINGS IN THE SPACE

Miloš Arsenović, Vesna Manojlović and Miodrag Mateljević

University of Belgrade, Faculty of Mathematics
Studentski Trg 16, Belgrade, Serbia; arsenovic@matf.bg.ac.yu

University of Belgrade, Faculty of Organizational Sciences
Jove Ilica 154, Belgrade, Serbia; vesnak@fon.bg.ac.yu

University of Belgrade, Faculty of Mathematics
Studentski Trg 16, Belgrade, Serbia; miodrag@matf.bg.ac.yu

Abstract. We obtain a sharp estimate of the derivatives of harmonic quasiconformal extension $u = P[\phi]$ of a Lipschitz map $\phi\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$. We also consider additional conditions which provide that u is Lipschitz on the unit ball; in particular, we give characterizations of Lipschitz continuity of u in the planar case and in the upper half space setting. We also answer a question posed by Martio in [OM] and extend this to the case of several variables.

1. Introduction and notations

Let $\mathbf{B} = \mathbf{B}^n = \{x \in \mathbf{R}^n \colon |x| < 1\}$ and $\mathbf{S} = \mathbf{S}^{n-1}$ denote the unit ball and the unit sphere in $\mathbf{R}^n$ respectively. We write $\mathbf{U}$ and $\mathbf{T}$ instead of $\mathbf{B}^2$ and $\mathbf{S}^1$ respectively; for $r > 0$, let $\mathbf{B}_r = \{x \colon |x| < r\}$ and $\mathbf{S}_r = \{x \colon |x| = r\}$.

Let $D \subset \mathbf{R}^n$ and $0 < \alpha \le 1$. The vector space of all functions $f\colon D \to \mathbf{R}^m$ satisfying the following condition: there is a constant $L = L_f$ such that $|f(x) - f(y)| \le L|x-y|^\alpha$ for all $x, y \in D$ is denoted by $\mathrm{Lip}(\alpha, D)$, or simply $\mathrm{Lip}\,\alpha$.

For $0 < \alpha < 1$ we say that $f \in \mathrm{Lip}\,\alpha$ is Hölder continuous on D with exponent α; for $\alpha = 1$ we write Lip instead of Lip 1 and we say that $f \in \mathrm{Lip}$ is Lipschitz continuous on D with multiplicative (Lipschitz) constant $L = L_f$. Let $\Lambda_\alpha(D) = \Lambda_\alpha$ be the Banach space of all bounded Hölder continuous functions $f\colon D \to \mathbf{R}^m$ with norm

$$\|f\|_\alpha = \sup_{x \in D} |f(x)| + \sup_{x,y \in D} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

It is known, even for $n = 2$, that Lipschitz continuity of $\phi\colon \mathbf{T} \to \mathbf{C}$, does not imply Lipschitz continuity of $u = P[\phi]$. In fact $u = P[\phi]$ is Lipschitz continuous iff the Hilbert transform of $\psi(\theta) = \frac{d}{d\theta}\phi(e^{i\theta})$ (which is defined almost everywhere and bounded since ϕ is Lipschitz) is also in $L^\infty(\mathbf{T})$. This result is implicitly contained in [Z], see also Theorem 2.4 below. The same theorem gives additional characterizations of Lipschitz continuity of u in terms of the Cauchy transform of ψ. A similar characterization, in terms of the Riesz transforms, is given in the setting of the upper half space $\mathbf{R}^{n+1}_+ = \{(x, y) \colon x \in \mathbf{R}^n, y > 0\}$ in Theorem 3.2. In particular, $f \in C^{1,\alpha}(\mathbf{R}^n)$

doi:10.5186/aasfm.2010.3524
2000 Mathematics Subject Classification: Primary 30C80, 30C62; Secondary 30C55, 30H05.
Key words: Lipschitz-type spaces, harmonic mappings, quasiregular mappings.
Research partially supported by MNTRS, Serbia, Grant No. 144 020.

implies $u = P[f] \in C^{1,\alpha}(\mathbf{R}^{n+1}_+)$. Here, for any $n \geq 2$,

$$P[\phi](x) = \int_{S^{n-1}} P(x,\xi)\phi(\xi)\, d\sigma(\xi), \quad x \in \mathbf{B}^n,$$

where $P(x,\xi) = \frac{1-|x|^2}{|x-\xi|^n}$ is the Poisson kernel for the unit ball $\mathbf{B}^n$, $d\sigma$ is the normalized surface measure on the unit sphere $\mathbf{S}^{n-1}$ and $\phi\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$ is a continuous mapping. The corresponding formula for the upper half space is

$$P[\phi](x,y) = \int_{\mathbf{R}^n} P(x-t,y)\phi(t)\, dt,$$

where

$$P(x,y) = c_n \frac{y}{(|x|^2+y^2)^{n+1/2}}, \quad c_n = \Gamma(\frac{n+1}{2})\pi^{-(n+1)/2},$$

is the Poisson kernel for the upper half space. The Riesz transforms R_j, $1 \leq j \leq n$, in $\mathbf{R}^n$ are defined by principal value integrals

$$R_j f(x) = c_n \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy,$$

they are bounded on $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ and $\Lambda_\alpha(\mathbf{R}^n)$ $(0 < \alpha < 1)$ spaces. It is important to note that these operators are not bounded on $L^1(\mathbf{R}^n)$, $L^\infty(\mathbf{R}^n)$ and $\Lambda_1(\mathbf{R}^n)$. We refer to [St] for a detailed discussion of these results in the context of singular integral operators.

Similar results hold in the $\mathbf{S}^{n-1}$ setting. Indeed, Hölder continuity of $\phi\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$ with exponent α, $0 < \alpha < 1$, implies Hölder continuity of its harmonic extension $u = P[\phi]$, see [Dy], [NO]. In the case $n = 2$ it is a classical result, following from Privalov's theorem (see [Z]).

In Section 3, using the maximum principle for harmonic functions, we prove:

Claim 1. If $\phi\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$ is Lipschitz continuous with Lipschitz constant $L_\phi = L$, then $u = P[\phi]$ is Lipschitz continuous with constant $L_u = L/r$ on the spheres $\mathbf{S}_r$, $0 < r < 1$.

In the case $n = 2$, using Schwarz lemma for harmonic functions, we prove an estimate $|\partial_\theta h(z)| \leq \frac{4}{\pi} L|z|$ (see Theorem 2.1).

Harmonic quasiconformal mappings were first studied by Martio in [OM]. Now it is a very active area of investigation (see [K3]). The following theorem has recently been proved in [AKM]:

Theorem 1.1. *Assume $\phi\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$ satisfies a Lipschitz condition*

$$|\phi(\xi) - \phi(\eta)| \leq L|\xi - \eta|, \quad \xi, \eta \in \mathbf{S}^{n-1},$$

and assume its harmonic extension $u = P[\phi]\colon \mathbf{B}^n \to \mathbf{R}^n$ is K-quasiregular. Then

$$|u(x) - u(y)| \leq C'|x-y|, \quad x, y \in \mathbf{B}^n,$$

where C' depends on L, K and n only.

Kalaj obtained a related result, but under additional assumption of $C^{1,\alpha}$ regularity of ϕ, see [K1]. In the case $n = 2$ this assumption (without hypothesis that u is K-quasiregular) implies that partial derivatives of u are Hölder continuous and, in particular, that u is Lipschitz on $\mathbf{U}$ (see Theorem 2.3).

The proof of Theorem 1.1 was based on estimates of the gradient of the Poisson integral kernel and did not yield sharp bounds on C'. We give another proof of this result, based on application of the maximum principle to a subharmonic function $A(x,a) = \sum_{\nu=1}^{n} |dh_\nu(x)a|^2$ and on Claim 1, where a is a unit vector. Using this approach we obtain $C' = KL$ (a dimension-free estimate).

2. The planar case

In the planar case we use the notation $z = re^{i\theta}$. If h is a function of variable z, we consider also h as a function of variables (r,θ) (polar coordinates). Also, for $f\colon \mathbf{T} \to \mathbf{C}$, we define $\hat{f}$ on $[0,2\pi]$ by $\hat{f}(t) = f(e^{it})$.

The following fact will be used below: if h is harmonic in $\mathbf{U}$, then $r\partial_r h$ is the harmonic conjugate of $\partial_\theta h$.

We refer the reader to [Du] for an excellent exposition on harmonic mappings in the plane, see also [BH].

It is known that Privalov theorem for harmonic functions with C^α boundary values, $0 < \alpha < 1$, fails for $\alpha = 1$. The next theorem deals with the case $\alpha = 1$ and explains that this failure is due to the loss of control of the radial derivative, see also [K].

Theorem 2.1. *Suppose that h is a harmonic mapping from $\mathbf{U}$ continuous on $\overline{\mathbf{U}}$. Then the following conditions are equivalent:*

a)
$$|h(e^{i\theta_1}) - h(e^{i\theta_2})| \le m|\theta_1 - \theta_2|. \tag{1}$$

b)
$$|h'(z)T| \le M \tag{2}$$
for every $z \in \mathbf{U}$ and unit vector $T = ie^{i\theta}$ which is tangent to the circle $\mathbf{S}_r$ at $z = re^{i\theta}$.

c) *$\partial_\theta h$ is bounded on $\mathbf{U}$.*

a) *implies* b) *with constant $M = \frac{4}{\pi}m$.* b) *implies* a) *with constant $m = M$.*

Proof. Suppose that a) holds. Then $|\frac{d}{d\theta}h(e^{i\theta}| \le m$ a.e. Since $\hat{h}$ is absolutely continuous on $[0,2\pi]$ and $\partial_\theta P_r(\theta - t) = -\partial_t P_r(\theta - t)$, using integration by parts, we have

$$\partial_\theta h(z) = -\frac{1}{2\pi}\int_0^{2\pi} \partial_t P_r(\theta - t)\, h(e^{it})\, dt = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\, d\hat{h} \ = P[\hat{h}'].$$

Therefore, by Proposition 3.2, $|\partial_\theta h(z)| \le m$, $z \in \mathbf{U}$. Since $\partial_\theta h(0) = 0$, by the harmonic version of Schwarz lemma, we find $|h'_\theta(z)| \le \frac{4}{\pi}m|z|$.

Now the estimate (2), with $M = \frac{4}{\pi}m$, follows easily. The remaining straightforward implications are left to the reader. □

An easy corollary of the above theorem is the following result:

Proposition 2.1. *Suppose that h is a harmonic quasiregular map in $\mathbf{U}$. Then the following conditions are equivalent:*

(1) *h is Lipschitz continuous on $\mathbf{U}$.*
(2) *h has continuous extension to $\overline{\mathbf{U}}$ which belongs to* Lip *on $\mathbf{T}$.*
(3) $\operatorname{grad} h$ *is bounded on $\mathbf{U}$, i.e.* $|\operatorname{grad} h(z)| \le A$, $z \in \mathbf{U}$.

For a function f defined on $\mathbf{U}$, we define

$$f_*(\theta) = f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

whenever the limit exists.

If $\psi \in L^1[0, 2\pi]$, the Cauchy transform $C[\psi]$ of ψ is defined by

$$C[\psi](z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\psi(t)e^{it}}{e^{it} - z}\, dt,$$

and the Hilbert transform of ψ is defined by

$$H\psi(\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\psi(\theta - t)\, dt}{\tan(t/2)},$$

where the integral is interpreted in the principal value sense.

In [OM] the following situation was considered: u is a harmonic function in the unit disc which assumes continuous boundary values f on $\mathbf{T}$, u_r and u_θ are derivatives of u with respect to r and θ. A question posed in that paper is: find necessary and sufficient conditions ensuring that the limits $\lim_{z\to\zeta} u_r(z) = \phi(\zeta)$ and $\lim_{z\to\zeta} u_\theta(z) = \psi(\zeta)$ exist at each boundary point $\zeta \in \mathbf{T}$. The following proposition answers that question, a related problem in higher dimensions is discussed in the next section.

Proposition 2.2. *In the above situation, both limits* $\lim_{z\to\zeta} u_r(z) = \phi(\zeta)$ *and* $\lim_{z\to\zeta} u_\theta(z) = \psi(\zeta)$ *exist at each boundary point* $\zeta \in \mathbf{T}$ *if and only if* $\hat{f}(t) = f(e^{it})$ *is continuously differentiable and* $H[\hat{f}']$ *is continuous.*

Proof. Assume that the two limits exist at each boundary point, then they define continuous functions $\psi(t) = \lim_{z\to e^{it}} u_\theta(z)$ and $\phi(t) = \lim_{z\to e^{it}} u_r(z)$. Therefore $u_\theta(re^{it})$ converges uniformly over t as $r \to 1$, which shows that $\hat{f}$ is a C^1 function and $u_\theta(re^{it}) \rightrightarrows \hat{f}'(t)$ as $r \to 1$. Similarly, $u_r(re^{it})$ converges uniformly over t as $r \to 1$ to a continuous function $g(t)$, hence $ru_r(re^{it}) \rightrightarrows g(t)$ as $r \to 1$. However, ru_r is the harmonic conjugate of u_θ and therefore the corresponding boundary functions are related by the Hilbert transform: $H[\hat{f}'] = g$.

Conversely, if f is C^1 and $H[\hat{f}'] = g$ is continuous, then the harmonic extension of $\hat{f}'$ is equal to u_θ and the harmonic extension of g is the harmonic conjugate of u_θ, that is, ru_r. Hence both u_θ and ru_r, and therefore u_r as well, have continuous extension to the boundary. □

In fact, in [OM], boundary functions f of the form $\hat{f}(t) = f(e^{it}) = e^{i\chi(t)}$ were considered, where χ is a continuous increasing function on $\mathbf{R}$ such that $\chi(t + 2\pi) = \chi(t) + 2\pi$ and the characterization problem was posed in terms of the function χ.

Theorem 2.2. *In the above situation, the limits* $\lim_{z\to\zeta} u_r(z) = \phi(\zeta)$ *and* $\lim_{z\to\zeta} u_\theta(z) = \psi(\zeta)$ *exist at each boundary point* $\zeta \in \mathbf{T}$ *if and only if* $\chi(t)$ *is continuously differentiable and* $H[\chi']$ *is continuous.*

Proof. Since $\hat{f}$ is C^1 if and only if χ is C^1, in view of the above proposition it suffices to prove the following statement: if χ is C^1, then $H[\hat{f}']$ is continuous if and only if $H[\chi']$ is continuous. Indeed, we have

$$H(\hat{f}')(\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\hat{f}'(\theta - t)\, dt}{\tan(t/2)} = -\frac{1}{\pi}\int_{+0}^{\pi} \frac{\hat{f}(\theta + t) + \hat{f}(\theta - t) - 2\hat{f}(\theta)}{2\sin^2(t/2)}\, dt$$

almost everywhere and therefore

$$e^{-i\chi(\theta)} H(\hat{f}')(\theta) = -\frac{1}{\pi}\int_0^{\pi} \frac{e^{i(\chi(\theta+t)-\chi(\theta))} + e^{i(\chi(\theta-t)-\chi(\theta))} - 2}{2\sin^2(t/2)}\, dt.$$

Define $S(\theta,t) = \sum_{k=2}^{\infty} a_k(\theta,t)$, where

$$a_k(\theta,t) = i^k \frac{(\chi(\theta+t)-\chi(\theta))^k + (\chi(\theta-t)-\chi(\theta))^k}{k!\, \sin^2(t/2)}.$$

Each of the functions $a_k(\theta,t)$, $k \geq 2$, is continuous and there is a constant C independent of k such that $|a_k(\theta,t)| \leq C^{k-2}/k!$ for $k \geq 2$. Therefore, S is a continuous function and $\int_0^{\pi} S(\theta,t)\, dt$ is a continuous function of θ.

Hence, using $e^{iu} = 1 + iu + E(u)$, where $E(u) = \sum_{k=2}^{\infty} (iu)^k/k!$, we find

$$e^{-i\chi(\theta)} H(\hat{f}')(\theta) = i\, H(\chi')(\theta) + R(\theta),$$

where R is a continuous function. Hence, $H(\hat{f}')$ is continuous if and only if $H(\chi')$ is continuous. □

Suppose that ϕ is Lipschitz on $\mathbf{T}$ and $h = P[\phi]$. Then $\partial_\theta h$ is bounded on $\mathbf{U}$ and $\partial_r h \in H^p$, $0 < p < \infty$; hence $(\partial_r h)^*$ exists a.e. on $\mathbf{T}$. Using a routine argument one can show that $\partial_r h(e^{it})$ exists a.e. and $(\partial_r h)^*(e^{it}) = \partial_r h(e^{it})$ a.e. on $\mathbf{T}$, where

$$\partial_r h(e^{it}) = \lim_{r\to 1-0} \frac{h^*(e^{it}) - h(re^{it})}{1-r}.$$

We say that $\phi \in C^{1,\alpha}(\mathbf{T})$, $0 < \alpha < 1$, if $\hat{\phi}'$ belongs to $\operatorname{Lip}\alpha$ on $[0, 2\pi]$.

In the next two theorems we use the following representation of a complex valued harmonic function h on $\mathbf{U}$: $h = f + \bar{g}$, where f and g are analytic. Note that f and g are unique, up to an additive constant.

Theorem 2.3. *If $\phi \in C^{1,\alpha}(\mathbf{T})$, $0 < \alpha < 1$, and $h = P[\phi]$, then*

a) *$\partial_\theta h$ and $\partial_r h$ belong to $\operatorname{Lip}\alpha$ on U, and*

b) *$|f''(z)| + |g''(z)| = O(1-r)^{\alpha-1}$, $z \in \mathbf{U}$, where $r = |z|$.*

In particular, h is Lipschitz on $\mathbf{U}$.

Proof. Since $\hat{\phi}$ is absolutely continuous on $[0, 2\pi]$, we find

$$\partial_\theta h(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\, d\hat{\phi}(t) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\, \hat{\phi}'(t)\, dt.$$

Hence, by Privalov's theorem, $\partial_\theta h$ belongs to $\operatorname{Lip}\alpha$ on $\mathbf{U}$. Since Hölder continuity of a harmonic function implies Hölder continuity of its harmonic conjugate, we conclude that $\partial_r h$ belongs to $\operatorname{Lip}\alpha$ on $\mathbf{U}$; therefore f' and g' belong to $\operatorname{Lip}\alpha$ on $\mathbf{U}$ and we get b).

In particular, $\operatorname{grad} h$ is bounded on $\mathbf{U}$ and therefore h is Lipschitz on $\mathbf{U}$. □

Theorem 2.4. *Suppose that ϕ is Lipschitz on $\mathbf{T}$ and $h = P[\phi]$. Then the following conditions are equivalent:*

(1) *h is Lipschitz on $\mathbf{U}$.*

(2) *The Cauchy transform $C[\hat{\phi}']$ is bounded.*

(3) *The Cauchy transform $C[\overline{\hat{\phi}'}]$ is bounded.*

(4) *$(\partial_r h)^*$ is bounded on $\mathbf{T}$.*

(5) $\partial_r h(e^{it})$ *is bounded on* $\mathbf{T}$.
(6) $|f'(z)| + |g'(z)|$ *is bounded on* $\mathbf{U}$.

Proof. Since ϕ is Lipschitz on $\mathbf{T}$, ϕ is absolutely continuous, and then $C[\phi'](z) = izf'(z)$ and $C[\overline{\hat{\phi}'}](z) = izg'(z)$, where C is the Cauchy transform and $izf'(z)$ is the analytic part of $\partial_\theta h$.

Since, by Theorem 2.1, $\partial_\theta h(z) = i(zf'(z) - \overline{zg'(z)})$ is bounded on $\mathbf{U}$, we see at once that (2), (3) and (6) are equivalent.

If (3) holds, then $\partial_r h$ is bounded on $\mathbf{U}$. The rest of the proof is routine. □

3. Higher dimensions

Now we turn to the general case.

Let f be a vector-valued function defined in a neighborhood of a point $z \in \mathbf{R}^n$, differentiable at z. By $f'(z)$ we denote the linear operator $df(z)$ between the tangent spaces at z and $f(z)$.

For linearly independent $x, y \in \mathbf{R}^n$, we denote by $L(x, y)$ the plane defined by 0, x and y. We can choose an orthonormal base $e_1, \dots, e_n$ such that $L(x, y) = L(e_1, e_2)$. For $z = (z_1, \dots, z_n) = \sum_{k=1}^n z_k e_k$ define $Pz = z_1e_1 + z_2e_2$ and $Qz = z_3e_3 + \cdots + z_ne_n$. If α is the oriented angle between x and y, we define the rotation $R = R_{x,y}$ by $Rz = (e^{i\alpha}Pz, Qz)$. Hence R is in the orthogonal group $O(n)$, acts as the identity map on the orthocomplement of $L(x, y)$ and, in the case $|x| = |y|$, maps x to y.

For $x \in \mathbf{R}^n$, $x \neq 0$, we set $x^* = \frac{x}{|x|} \in \mathbf{S}$. Note that $|Rz - z| = |e^{i\alpha}Pz - Pz| = |Pz||1 - e^{i\alpha}| \le |x^* - y^*|$ for $z \in \overline{\mathbf{B}}$; and if $|x| = |y|$, then $Rx = y$ and $Rx^* = y^*$. Thus

$$\max\{|Rz - z| \colon z \in \mathbf{S}\} \le |x^* - Rx^*| = |x^* - y^*|. \tag{3.1}$$

Since the Laplacian commutes with orthogonal transformations [ABR, pp. 3–4], we have:

Proposition 3.1. *If* $h\colon \mathbf{B}^n \to \mathbf{R}^n$ *is harmonic on* $\mathbf{B}^n$, *then* $h \circ R$ *is harmonic.*

Proposition 3.2. *Suppose that* h *is a harmonic mapping from* $\mathbf{B}^n$ *continuous on* $\overline{\mathbf{B}}^n$ *and* $M = \max\{|h(t)| \colon t \in \mathbf{S}\}$. *Then* $|h(t)| \le M$ *for* $|t| \le 1$.

A proof can be based on the Poisson representation, or, alternatively, reduced (using scalar product) to the classical real valued case.

We will also use a version of Harnack's inequality (see [MV]): Let $B = B(a; r) \subset \mathbf{R}^n$ be a ball. Suppose $h\colon B \to \mathbf{R}^n$ is a vector valued harmonic mapping on B and $M_a = \sup\{|h(y) - h(a)| \colon y \in B\}$. Then

$$r|h'(a)| \le nM_a. \tag{3.2}$$

Conjecture 1. Let D be a domain in $\mathbf{R}^n$ with C^1 (or C^∞) boundary, let $y_0 \in D$ and let $g(x) = g(x, y_0)$ be the Green's function for D. Set $S_c = \{x \in D \colon g(x) = c\}$. Suppose that $h\colon D \to \mathbf{R}^m$ is a harmonic mapping which is continuous on $\overline{D}$. The following conditions are equivalent:

a) h is Lipschitz on ∂D.
b)
$$|h'(x)T| \le M$$
for every $x \in D$ and unit vector T which is tangent to S_c, $c > 0$, at x.

We prove this conjecture for the unit ball, taking $y_0 = 0$.

Theorem 3.1. *Suppose that* $h\colon \overline{\mathbf{B}}^n \to \mathbf{R}^n$ *is harmonic on* $\mathbf{B}^n$ *and continuous on* $\overline{\mathbf{B}}^n$. *Then the following conditions are equivalent:*

a)

$$|h(x) - h(y)| \le L|x-y|, \ x,y \in \mathbf{S}. \tag{1'}$$

b)

$$|h'(x)T| \le M \tag{2'}$$

for every $x \in \mathbf{B}^n$ *and unit vector* T *which is tangent on* $\mathbf{S}_r$, *where* $r = |x|$.

If we suppose, in addition, that h *is* K*-quasiregular mapping, then*

c)

$$|h'(x)| \le K\,L \tag{3'}$$

for every $x \in \mathbf{B}^n$.

Proof. It is clear that b) implies a), with constant $L = M$.

Suppose that a) holds. Let $x_0 \in \mathbf{S}$ be fixed. Then $|h(x) - h(x_0)| \le C_1 = 2L$ on $\mathbf{S}$ and by Poisson representation $|h(x) - h(x_0)| \le C_1 = 2L$ on $\mathbf{B}$. Using translation, we can suppose that $h(x_0) = 0$. Hence $|h(x) - h(a)| \le C_2 = 4L$ for $x, a \in \mathbf{B}$.

If $|x| \le 1/2$, then an application of (3.2) on the ball $\mathbf{B}(x; 1/2)$ gives $|h'(x)| \le 2nC_2$. Hence there is a constant $C_3 = 2nC_2 = 8nL$ such that

$$|h(x) - h(y)| \le C_3|x-y| \tag{3.3}$$

for every $x, y \in \mathbf{B}_{1/2}$.

Let us prove that

$$|h(x) - h(y)| \le L|x^* - y^*| = L_r|x-y|, \quad |x| = |y| = r,$$

where $L_r = L/r$.

Let $R = R_{x,y}$ be the rotation described above which maps x to y. Note that $\max\{|Rz - z|\colon z \in \mathbf{S}\} \le |x^* - y^*|$. By Proposition 3.1, the function $h(z) - h(Rz)$ is harmonic in z. By hypothesis a), $|h(z) - h(Rz)| \le L|Rz - z|$, $z \in \mathbf{S}$. Hence, by (3.1), $\max\{|h(z) - h(Rz)|\colon z \in \mathbf{S}\} \le L|x^* - Rx^*|$. Now applying Proposition 3.2 (the maximum principle), we conclude that $|h(x) - h(Rx)| \le L_r|x - Rx|$. Thus $|h(x) - h(y)| \le L_r|x-y|$ whenever $|x| = |y| = r < 1$. Clearly this proves the following estimate:

$$|h'(x)T| \le L_r \tag{3.4}$$

for every $x \in \mathbf{B}^n$ and unit vector T which is tangent to $\mathbf{S}_r$, where $r = |x|$. In particular, for $r \ge 1/2$, $|h'(x)T| \le 2L$. By (3.3), we can choose $M = 8nL$.

Now we prove c). Let $a \in \mathbf{S}^{n-1}$ be a fixed unit vector. Then the function $A(x,a) = |dh(x)a|^2 = \sum_{\nu=1}^n |dh_\nu(x)a|^2$ is subharmonic in $x \in \mathbf{B}$. Using estimate (3.4) and quasiregularity of h we obtain $|dh(x)a| \le KL/\rho$ on $\mathbf{S}_\rho$, $0 < \rho < 1$. Now the maximum principle for subharmonic functions gives, as $\rho \to 1$, $|dh(x)a| \le KL$ on $\mathbf{B}$, and since a is an arbitrary unit vector we conclude $|h'(x)| \le KL$. □

One consequence of the tangential estimate (3.4) is:

Proposition 3.3. *Suppose that* $\phi\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$ *is Lipschitz. Let* $u = P[\phi]$. *Then the following conditions are equivalent:*

(1) $u = P[\phi]\colon \mathbf{B}^n \to \mathbf{R}^n$ *is Lipschitz on* $\mathbf{B}^n$.
(2) *The radial derivative of* u *is bounded on* $\mathbf{B}^n$.
(3) $\operatorname{grad} u$ *is bounded on* $\mathbf{B}^n$.

Next we consider the upper half space $\mathbf{R}^{n+1}_+$ setting. First, we note that a harmonic map $u\colon \mathbf{R}^{n+1}_+ \to \mathbf{R}^{n+1}$ is bounded and extends continuously to the boundary if and only if $u = P[f]$ for some bounded continuous map $f\colon \mathbf{R}^n \to \mathbf{R}^{n+1}$. In this case f is Lipschitz continuous if and only if the partial derivatives $\partial_j u$, $1 \le j \le n$, are bounded on $\mathbf{R}^{n+1}_+$. This is, of course, a necessary condition for Lipschitz continuity of u. To get a necessary and sufficient condition, one has to ensure that $\partial u/\partial y$ is bounded as well. Let $\mathscr{F} f(\xi)$ be the Fourier transform of f in the sense of distributions, i.e., $\mathscr{F} f$ is in the space $\mathscr{S}'$ of tempered distributions. Then $i\xi_j \mathscr{F} f(\xi)$ is the Fourier transform of $\partial_j f$ for $1 \le j \le n$. Also, the Fourier transform of $u(x,y)$ for a fixed $y > 0$ is $e^{-y|\xi|}\mathscr{F} f(\xi)$ and therefore the Fourier transform of $\partial_y u(x,y)$ is $-|\xi|e^{-y|\xi|}\mathscr{F} f(\xi)$. Hence, taking the limit $y \to 0$ in $\mathscr{S}'$, we see that the boundary values g of $\partial u/\partial y$ satisfy $\mathscr{F} g(\xi) = -|\xi|\mathscr{F} f(\xi)$. Therefore, $\mathscr{F}(\partial_j f)(\xi) = -i\xi_i/|\xi|\mathscr{F} g(\xi)$ which means that $\partial_j f = R_j g$, where R_j denotes the Riesz transform. We can summarize the above discussion in the following theorem.

Theorem 3.2. *A harmonic map $u\colon \mathbf{R}^{n+1}_+ \to \mathbf{R}^{n+1}$ is bounded and Lipschitz continuous if and only if $u = P[f]$, where f is bounded and Lipschitz continuous on the boundary $\mathbf{R}^n$, $\phi_j = \partial f/\partial x_j$ are in $L^\infty(\mathbf{R}^n)$ for all $1 \le j \le n$ and for some (equivalently all) j the function ϕ_j is the R_j transform of a function in $L^\infty(\mathbf{R}^n)$.*

Note that the theorem remains valid, with essentially the same proof, if one replaces "bounded and Lipschitz continuous" with "bounded with continuous and bounded partial derivatives" and $L^\infty(\mathbf{R}^n)$ with $BC(\mathbf{R}^n)$ (the space of continuous and bounded functions on $\mathbf{R}^n$). This extends Proposition 2.2 to the multidimensional case.

It is easy to derive sufficient conditions from the above result: since the Riesz transforms R_j preserve Λ_α spaces, $0 < \alpha < 1$, any bounded function with partial derivatives in Λ_α extends to a Lipschitz continuous harmonic function in the upper half space, in fact that extension is in $C^{1,\alpha}(\mathbf{R}^{n+1}_+)$.

References

[Ahl] Ahlfors, L.: Möbius transformation in several dimensions. - School of Mathematics, University of Minnesota, 1981.

[AKM] Arsenović, M., V. Kojić, and M. Mateljević On Lipschitz continuity of harmonic quasiregural mappings on the unit ball in $\mathbf{R}^n$. - Ann. Acad. Sci. Fenn. Math. 33, 2008, 315–318.

[ABR] Axler, S., P. Bourdon, and W. Ramey: Harmonic function theory. - Springer-Verlag, New York, 1992.

[BH] Bshouty, D., and W. Hengartner: Univalent harmonic mappings in the plane. - In: Handbook of Complex Analysis: Geometric Function Theory, Volume 2, edited by R. Kühnau, 2005, 479–506.

[Du] Duren, P.: Harmonic mappings in the plane. - Cambridge Univ. Press, 2004.

[Dy] Dyakonov, K. M.: Holomorphic functions and quasiconformal mappings with smooth moduli. - Adv. Math. 187:1, 2004, 146–172.

[IM] Iwaniec, T., and G. Martin: Geometric function theory and nonlinear analysis. - Syracuse and Auckland, 2000.

[K] Kalaj, D.: Harmonijske funkcije i kvazikonformna preslikavanja (Harmonic functions and quasiconformal mappings). - Master thesis, 1998 (in Serbian).

[K1] Kalaj, D.: On harmonic quasiconformal self-mappings of the unit ball. - Ann. Acad. Sci. Fenn. Math. 33, 2008, 261–271.

[K2] Kalaj, D.: Quasiconformal harmonic mapping between Jordan domains. - Math. Z. 260:2, 2008, 237–252.

[K3] Kalaj, D.: Lipschitz spaces and harmonic mappings. - Ann. Acad. Sci. Fenn. Math. 34:2, 2009, 475–485.

[KM] Kalaj, D., and M. Mateljević: Inner estimate and quasiconformal harmonic maps between smooth domains. - J. Anal. Math. 100, 2006, 117–132.

[MV] Manojlović, V.: Bi-Lipschicity of quasiconformal harmonic mappings in the plane. - Filomat 23:1, 2009, 85–89.

[OM] Martio, O.: On harmonic quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 425, 1968, 3–10.

[M] Mateljević, M.: Distortion of harmonic functions and harmonic quasiconformal quasi-isometry. - Rev. Roumaine Math. Pures Appl. 51:5-6, 2006, 711–722.

[M1] Mateljević, M.: Lipschitz-type spaces and quasiregular harmonic mappings in the space and applications. - Manuscript, 2007.

[M2] Mateljević, M.: On quasiconformal harmonic mappings. - Manuscript, 2006.

[MV] Mateljević, and M. Vuorinen: On harmonic quasiconformal quasi-isometries. - arXiv: 0709.4546v1.

[NO] Nolder, C. A., and D. M. Oberlin: Modulus of continuity and a Hardy–Littlewood theorem. - Lecture Notes in Math. 1351, 1988, 265–272.

[Ri] Rickman, S.: Quasiregular mappings. - Springer-Verlag, Berlin Heidelberg, 1993.

[St] Stein, E.: Singular integrals and differentiability properties of functions. - Princeton Univ. Press, 1970.

[Z] Zygmund, A.: Trigonometrical series. - Chelsea Publishing Co., 2nd ed., New York, 1952.

Received 8 May 2009