Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 35, 2010, 633–640

A POTENTIAL THEORY APPROACH TO THE EQUATION $-\Delta u = |\nabla u|^2$

Anna Tuhola-Kujanpää

University of Jyväskylä, Department of Mathematics and Statistics
P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland; anna.tuhola-kujanpaa@jyu.fi

Abstract. In this paper we show that if

$$\begin{cases} -\Delta u = |\nabla u|^2 \ \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{1}$$

then

$$-\Delta(e^u - 1) = \mu \ \text{ in } \Omega, \tag{2}$$

when $\mu \perp \mathrm{cap}_2$, and conversely.

1. Introduction

In this paper we show that by a change of variable we can transform a Laplace equation with quadratic growth in the gradient to one with a singular measure on the right hand side. More precisely we have:

1.1. Theorem. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Then u is a solution of*

$$\begin{cases} -\Delta u = |\nabla u|^2 \ \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3}$$

if and only if $e^{\alpha u/2} - 1 \in W_0^{1,2}(\Omega)$ for all $0 < \alpha < 1$ and there exists a positive Radon measure μ such that $e^u - 1$ is a weak solution of

$$-\Delta(e^u - 1) = \mu \ \text{ in } \Omega \tag{4}$$

and $\mu \perp \mathrm{cap}_2$.

The equation (3) is an analytic equation that does not allow any other bounded solutions but the constant 0. Here we characterize all possible solutions.

A similar result can be found in [1] and its corrigendum [2]. However, our proof extends to a case where μ is an arbitrary Radon measure, not necessarily bounded. Our approach is based on a very different technique, namely potential theory and it relies in partial on the Riesz decomposition theorem. We also employ renormalized solutions discussed in [10].

In the proof of Theorem 1.1 we also need the uniqueness of harmonic functions in $W_0^{1,p}(\Omega)$. This is an interesting result of its own and Section 2 is devoted to its proof and comments. To show that our assumptions on the domain Ω are relevant, we include a counterexample by Hajłasz [13] and construct another counterexample in a domain with a very irregular boundary. Recently Brezis [4] and Jin et al. [15] have studied similar problems locally without considering the regularity of the domain.

---

doi:10.5186/aasfm.2010.3540
2000 Mathematics Subject Classification: Primary 35J05, 35J25, 31B20, 31B05.
Key words: Elliptic equations, singular solutions, uniqueness.

Problems with equations similar to (3) have been widely studied. See for instance [6],[8], [7], [9], [11] and [16].

2. Uniqueness of harmonic functions in $W^{1,p}(\Omega)$

In $W^{1,2}(\Omega)$ the uniqueness of harmonic functions is a familiar fact: for fixed $v \in W^{1,2}(\Omega)$ there is a unique harmonic function u such that $u - v \in W_0^{1,2}(\Omega)$. In $W^{1,p}(\Omega)$, when $1 < p < 2$, it is difficult to locate the corresponding fact from the literature.

In Theorem 2.1 we find a sufficient condition for the uniqueness to hold in a domain Ω: the smoothness of the domain is expressed in terms of the integrability of the gradient of the Green function. (For more information of the Green function, see for instance [5].) This Theorem 2.1 will be later applied in the proof of Theorem 1.1 in a smooth domain. However, a bounded domain with $C^{1,\alpha}$ boundary for some $\alpha > 0$ is regular enough to satisfy the assumptions.

By the space $W_0^{1,p}(\Omega)$ we mean the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

2.1. Theorem. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and G the Green function associated with Ω. Suppose $v \geq 0$ is superharmonic $W_0^{1,p}(\Omega)$-function for some $p \geq 1$. If for some $x_0 \in \Omega$ there exists $K \subset\subset \Omega$ such that $\nabla_y G(x_0, y) \in L^{p/(p-1)}(\Omega \setminus K)$, when $p > 1$, or $\nabla_y G(x_0, y) \in L^\infty(\Omega \setminus K)$, when $p = 1$, then the greatest harmonic minorant h of v is 0.*

Proof. Notice first that by the minimum principle either $h < v$ in Ω or v itself is harmonic: if $h(x) = v(x)$ for some $x \in \Omega$, then for the non-negative superharmonic function $v - h$ we have $(v - h)(x) = 0$ and so $v - h$ attains its minimum in Ω. Hence $v - h$ is a constant function and it follows that v is harmonic.

Assume first that $h < v$. Take a sequence $\varphi_j \in C_0^\infty(\Omega)$ such that $\varphi_j \to v$ in $W^{1,p}(\Omega)$ and that for every compact $S \subset \Omega$ there exists $J \in \mathbf{N}$ such that $\varphi_j \geq h$ in S when $j > J$.

Fix $x_0 \in \Omega$ and denote $g(y) = G(x_0, y)$. Define $\Omega_t = \{y \in \Omega\colon g(y) > t\}$ for each $t > 0$. Denote the Green function of Ω_t by G_t and $g_t(y) = G_t(x_0, y)$. Observe that $g_t(y) = g(y) - t$, and hence $|\nabla g_t(y)| = |\nabla g(y)|$.

Denote the greatest harmonic minorant of φ_j in Ω_t by $h_{t,j}$. We have $h(x_0) \leq h_{t,j}(x_0)$ for all large j. Functions φ_j have compact supports and so it is justified to use the Green formula in $\Omega \setminus \Omega_t$ for φ_j and g. By the harmonicity of g near the boundary we have

$$
\begin{aligned}
h_{t,j}(x_0) &= \int_{\partial\Omega_t} \varphi_j \frac{\partial g_t}{\partial \nu}\, dS = \int_{\partial\Omega_t} \varphi_j \frac{\partial g}{\partial \nu}\, dS \\
&= \int_{\Omega\setminus\Omega_t} \nabla\varphi_j \cdot \nabla g\, dy + \int_{\Omega\setminus\Omega_t} \varphi_j\, \Delta g\, dy - \int_{\partial\Omega} \varphi_j \frac{\partial g}{\partial \nu}\, dS \qquad (5) \\
&= \int_{\Omega\setminus\Omega_t} \nabla\varphi_j \cdot \nabla g\, dy \to \int_{\Omega\setminus\Omega_t} \nabla v \cdot \nabla g\, dy,
\end{aligned}
$$

for when t is small enough, $\Omega \setminus \Omega_t \subset \Omega \setminus K$ and hence by Hölder's inequality

$$
\int_{\Omega\setminus\Omega_t} (\nabla\varphi_j - \nabla v) \cdot \nabla g\, dy \leq \left(\int_{\Omega\setminus\Omega_t} |\nabla\varphi_j - \nabla v|^p dy \right)^{\frac{1}{p}} \left(\int_{\Omega\setminus\Omega_t} |\nabla g|^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}} \to 0,
$$

when $j \to \infty$. This implies

$$\lim_{t\to 0}\Big(\lim_{j\to\infty} h_{t,j}(x_0)\Big) = \lim_{t\to 0}\Big(\lim_{j\to\infty}\int_{\Omega\setminus\Omega_t}\nabla\varphi_j\cdot\nabla g\,dy\Big) = \lim_{t\to 0}\int_{\Omega\setminus\Omega_t}\nabla v\cdot\nabla g\,dy = 0,$$

since $\nabla v\cdot\nabla g$ is integrable in $\Omega\setminus\Omega_t$ when t is small. Since $h(x)\le h_{t,j}(x)$, it follows that $h(x)=0$.

In the case v is harmonic, we can find a sequence $\varphi_j\in C_0^\infty(\Omega)$ such that $\varphi_j\to v$ in $W^{1,p}(\Omega)$ and φ_j converge to v locally uniformly. If we denote the greatest harmonic minorants of φ_j and v in Ω_t by $h_{t,j}$ and h_t respectively, we have by the uniform convergence on $\bar\Omega_t$ that $h_{t,j}(y)\to h_t(y)$ for all $y\in\Omega_t$.On the other hand, we know by the previous calculation (5) which is also valid in this case, that

$$h_{t,j}(x_0)\to\int_{\Omega\setminus\Omega_t}\nabla v\cdot\nabla g\,dy,$$

when $j\to\infty$. Hence

$$h_t(x_0)=\int_{\Omega\setminus\Omega_t}\nabla v\cdot\nabla g\,dy$$

and by the integrability of $\nabla v\cdot\nabla g$ in $\Omega\setminus\Omega_t$ when t is small, we obtain $h_t(x_0)\to 0$, when $t\to 0$, since $|\Omega\setminus\Omega_t|\to 0$. The result follows from the fact that $h_t(x_0)\ge h(x_0)$. □

In the proof above it is explicitely shown the following.

2.2. Corollary. *If Ω is as in Theorem 2.1 and $p\ge 1$, then the only harmonic function in $W_0^{1,p}(\Omega)$ is the zero function.*

As an immediate consequence we get the next corollary.

2.3. Corollary. *Suppose that $p\ge 1$. If Ω is as in Theorem 2.1 and $v\in W^{1,p}(\Omega)$, then there exists at most one harmonic function u such that $u-v\in W_0^{1,p}(\Omega)$.*

2.4. Remark. When $p\ge 2$ the previous Theorem is trivial, but also the assumptions of the Theorem are apparent: Let $x\in\Omega$ and denote $G_x(y)=G(x,y)$. Then the zero extension of G_x is subharmonic in $\mathbf{R}^n\setminus B(x,r)$ for all $r>0$, for G_x is harmonic in $\Omega\setminus B(x,r)$. Subharmonic functions belong to $W^{1,2}_{loc}(\mathbf{R}^n\setminus B(x,r))$. Since $2\ge p/(p-1)$, when $p\ge 2$, we have $\nabla G\in L^{p/(p-1)}(\Omega\setminus B(x,r))$ for all $r>0$.

2.5. Remark. Theorem 2.1 is not completely trivial. In [13] Hajłasz gives a counterexample in the case $1<p<\frac{4}{3}$: There exists a domain $\Omega\subset\mathbf{R}^2$ and a non-zero harmonic function $u\in W_0^{1,p}(\Omega)$. Here Ω is the image of set $D=\{z\in\mathbf{C}\colon |z-i|<1\}$ under mapping $z\mapsto z^2$. The domain Ω has one inward cusp and it satisfies the cone property. In the following we construct a counterexample in $\mathbf{R}^n$ for all $1<p<2$ with a domain far from simply connected.

2.6. Example. Let $1<p<2$. We can find a Cantor set $E\subset\mathbf{R}^n$ such that $\operatorname{cap}_2(E)>0$ and $\dim_{\mathscr{H}}(E)<n-p$ [3, Section 5.3]. Then $\operatorname{cap}_p(E)=0$. Take a ball $B\subset\mathbf{R}^n$ containing E and denote $\Omega=2B\setminus E$. Now the 2-potential $\hat R_E^1$ of E in $2B$ is harmonic in Ω, but not the zero function, since $\operatorname{cap}_2(E)>0$ [5, Theorem 5.3.4.(iii) and Lemma 5.3.3]. Clearly $\hat R_E^1\in W_0^{1,p}(\Omega)$ because $\operatorname{cap}_p(E)=0$ [14, Theorem 8.6].

3. Proof of Theorem 1.1 with some preparatory results

For the proof of the main theorem we need few auxiliary results. For them, denote

$$v_{\mu,\Omega}(x) = \int_\Omega G(x,y)\,d\mu(y),$$

where G is the Green function of Ω.

The p-capacity $\operatorname{cap}_p(A)$ for $1 < p \le N$ is defined in the following classical way: The p-capacity of a compact set $K \subset \Omega$ is first defined as

$$\operatorname{cap}_p(K) = \inf\{\int_\Omega |\nabla\varphi|^p dx \colon \varphi \in C_0^\infty(\Omega), \varphi(x) \ge 1 \text{ for all } x \in K\}.$$

The p-capacity of any open subset $U \subset \Omega$ is then defined by

$$\operatorname{cap}_p(U) = \sup\{\operatorname{cap}_p(K) \colon K \text{ compact}, K \subset U\}$$

Finally, the p-capacity of an arbitrary subset $A \subset \Omega$ is defined by

$$\operatorname{cap}_p(A) = \inf\{\operatorname{cap}_p(U) \colon U \text{ open}, A \subset U\}.$$

For the properties of the p-capacity, see [3].

3.1. Lemma. *For every Radon measure μ in Ω there exist unique Radon measures μ_0 and μ_s in Ω such that $\mu = \mu_0 + \mu_s$, $\mu_0 << \operatorname{cap}_2$ and $\mu_s \perp \operatorname{cap}_2$.*

Proof. See [12], Lemma 2.1. □

3.2. Lemma. *Suppose μ is a positive Radon measure in Ω and $\mu = \mu_0 + \mu_s$ as above in Lemma 3.1. Then $\mu_s\{v_{\mu,\Omega}(x) < \infty\} = 0$.*

Proof. Let $A \subset \Omega$ such that $\operatorname{cap}_2(A) = 0$ and $\mu_s(\Omega \setminus A) = 0$. If $\mu_s\{v_{\mu,\Omega}(x) < \infty\} > 0$, then $\mu_s\{v_{\mu,\Omega}(x) < k\} > 0$ for some $k > 0$. Let $K \subset \{v_{\mu,\Omega}(x) < k\} \cap A$ be compact. By an alternative definition of capacity [5, Theorem 5.5.5] we have

$$\begin{aligned} 0 &= \operatorname{cap}_2(K) \\ &= \sup\{\nu(K) \colon \nu \text{ positive measure, } \operatorname{spt}(\nu) \subset K, v_{\mu,\Omega}(x) < 1 \text{ for all } x \in K\} \\ &\ge \frac{1}{k}\mu\lfloor_K(K) \ge \frac{1}{k}\mu_s(K). \end{aligned}$$

It follows that $\mu_s\{v_{\mu,\Omega}(x) < k\} = 0$. This is a contradiction. □

Denote $T_k(f) = \min\{k, \max\{f, -k\}\}$ for all $k \ge 0$.

3.3. Lemma. *If $u \in W_0^{1,2}(\Omega)$ such that $-\Delta u = |\nabla u|^2$, then $e^{\frac{\alpha}{2}u} - 1 \in W_0^{1,2}(\Omega)$ for all $\alpha < 1$.*

Proof. In the view of the Sobolev inequality is enough to show the integrability of $|\nabla e^{\alpha u/2}|^2$, since the zero boundary values follow immediately from the zero boundary values of u.

Fix $k > 0$ and $0 < \alpha < 1$. Function $e^{T_k(u)} - 1 \in W_0^{1,2}(\Omega)$ and therefore it can be chosen as a test function in equation (3). So

$$\begin{aligned} \int_{\{u \le k\}} |\nabla u|^2 e^u dx &= \int_\Omega \nabla u \cdot \nabla e^{T_k(u)} dx = \int_\Omega |\nabla u|^2 (e^{T_k(u)} - 1) dx \\ &= \int_{\{u \le k\}} |\nabla u|^2 e^u dx + \int_{\{u > k\}} |\nabla u|^2 e^k dx - \int_\Omega |\nabla u|^2 dx, \end{aligned}$$

which gives

$$e^{-k}\int_\Omega |\nabla u|^2 dx = \int_{\{u>k\}} |\nabla u|^2 dx. \tag{6}$$

Also $e^{\alpha T_k(u)} - 1 \in W_0^{1,2}(\Omega)$, so it is a valid test function in equation (3). Hence we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 (e^{\alpha T_k(u)} - 1)dx &= \int_\Omega \nabla u \cdot \nabla (e^{\alpha T_k(u)} - 1)dx \\ &= \alpha \int_\Omega e^{\alpha T_k(u)} \nabla u \cdot \nabla T_k(u) dx,\end{aligned}$$

which together with equation (6) yields

$$\begin{aligned}(\alpha - 1)\int_{\{u\le k\}} e^{\alpha u}|\nabla u|^2 dx &= e^{\alpha k}\int_{\{u>k\}} |\nabla u|^2 dx - \int_\Omega |\nabla u|^2 dx \\ &= e^{\alpha k} e^{-k}\int_\Omega |\nabla u|^2 dx - \int_\Omega |\nabla u|^2 dx.\end{aligned}$$

By letting $k \to \infty$ we have

$$(1-\alpha)\int_\Omega |\nabla (e^{\alpha u/2} - 1)|^2 dx = \left(\frac{\alpha}{2}\right)^2 \int_\Omega |\nabla u|^2 dx < \infty. \tag{7}$$

Hence by the Sobolev inequality we have $e^{\frac{\alpha}{2}u} - 1 \in W_0^{1,2}(\Omega)$. □

3.4. Remark. Lemma 3.3 is sharp: Function $e^{\frac{u}{2}} - 1 \notin W_0^{1,2}(\Omega)$ unless $u \equiv 0$. This can be seen by letting $\alpha \to 1$ in equation (7). If $e^{\frac{u}{2}} - 1 \in W_0^{1,2}(\Omega)$, then the left hand side of the equation tends to zero making ∇u the zero function.

3.5. Lemma. *If $u \in W_0^{1,2}(\Omega)$ such that $-\Delta u = |\nabla u|^2$, then $e^{\alpha u} - 1 \in W^{1,1}(\Omega)$ and $e^{\alpha u} - 1$ is superharmonic for all $\alpha < 1$.*

Proof. To see that $e^{\alpha u} - 1 \in W^{1,1}(\Omega)$ we need to notice only, that by denoting $\nu = e^{\alpha u}dx$, a bounded Radon measure, we have

$$\begin{aligned}\int_\Omega |\nabla (e^{\alpha u} - 1)| dx = \alpha \int_\Omega |\nabla u| d\nu &\le c\left(\int_\Omega |\nabla u|^2 d\nu\right)^{1/2} \\ &= c\left(\int_\Omega |\nabla (e^{\alpha u/2} - 1)|^2 dx\right)^{1/2}\end{aligned}$$

which is finite by Lemma 3.3.

Function $e^{\alpha u} - 1 \in W^{1,1}(\Omega)$ is a supersolution for the equation $-\Delta v = 0$ in Ω for every $0 < \alpha < 1$: Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \ge 0$. Now $e^{\alpha T_k(u)}\varphi \in W_0^{1,2}(\Omega)$ and $e^{\alpha T_k(u)}\varphi \ge 0$ for every $k > 0$. By the dominated convergence theorem, valid here because of Lemma 3.3, we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{\alpha u}\varphi\, dx &= \lim_{k\to\infty}\int_\Omega |\nabla u|^2 e^{\alpha T_k(u)}\varphi\, dx = \lim_{k\to\infty}\int_\Omega \nabla u \cdot \nabla (e^{\alpha T_k(u)}\varphi)\, dx \\ &= \lim_{k\to\infty}\left(\int_\Omega e^{\alpha T_k(u)}\nabla u \cdot \nabla \varphi\, dx + \alpha\int_\Omega e^{\alpha T_k(u)}\varphi |\nabla T_k(u)|^2\, dx\right)\end{aligned}$$

$$= \int_\Omega e^{\alpha u} \nabla u \cdot \nabla \varphi \, dx + \alpha \int_\Omega e^{\alpha u} \varphi |\nabla u|^2 \, dx$$
$$= \frac{1}{\alpha} \int_\Omega \nabla e^{\alpha u} \cdot \nabla \varphi \, dx + \alpha \int_\Omega e^{\alpha u} \varphi |\nabla u|^2 \, dx$$

which implies

$$\int_\Omega \nabla (e^{\alpha u} - 1) \cdot \nabla \varphi \, dx = \alpha(1-\alpha) \int_\Omega e^{\alpha u} \varphi |\nabla u|^2 dx \geq 0.$$

So $e^{\alpha u} - 1$ is superharmonic. □

Now we have all the ingredients for the proof of the main result.

Proof of Theorem 1.1. Assume first that u is a solution of equation (3). Since $e^{\alpha u} - 1$ is superharmonic for all $0 < \alpha < 1$ (Lemma 3.5), we have by letting $\alpha \to 1$ that $e^u - 1$ is superharmonic [14, Lemma 7.3]. Consequently [14, Theorem 7.45] $\nabla(e^u - 1) \in L^q(\Omega)$ for all $q < n/(n-1)$ and hence $e^u - 1 \in W_0^{1,q}(\Omega)$.

Denote by μ the Riesz measure of function $e^u - 1$. Let $\varphi \in C_0^\infty(\Omega)$. Choose a C^∞-set $D \subset\subset \Omega$ such that $\operatorname{spt}(\varphi) \subset\subset D$. Then $\mu(D) < \infty$ and there is a positive function w, that solves the equation

$$\begin{cases} -\Delta w = \mu & \text{in } D, \\ \quad w = 0 & \text{on } \partial D \end{cases}$$

in the renormalized sense [10, Theorem 3.1]. By the Riesz decomposition theorem there exist harmonic minorants of $e^u - 1$ and w in D, h and h_w respectively, such that $e^u - 1 = v_{\mu,D} + h$ and $\omega = v_{\mu,D} + h_w$. However, by Theorem 2.1 we know that $h_w = 0$. Hence $w = v_{\mu,D}$ and in D we have $e^u - 1 = w + h$, where h is a non-negative harmonic function.

Let $k > 0$. We have $e^{-T_k(u)}\varphi \in W_0^{1,2}(D) \cap L^\infty(D)$, $\operatorname{spt}(e^{-T_k(u)}\varphi) \subset\subset D$ and $e^{-k}\varphi = e^{-T_k(u)}\varphi$ in $\{e^u > k+1\}$, $e^{-k}\varphi \in C_0^\infty(D)$. Since w is a solution in the renormalized sense and $\{w > k\} \subset \{e^u - 1 > k\}$, we have

$$\int_\Omega \nabla w \cdot \nabla \left(\frac{\varphi}{e^{T_k(u)}} \right) dx = \int_\Omega e^{-T_k(u)} \varphi \, d\mu.$$

By harmonicity $h \in W_{loc}^{1,2}(\Omega)$, and hence we have

$$\int_\Omega \nabla h \cdot \nabla \left(\frac{\varphi}{e^{T_k(u)}} \right) dx = 0.$$

Thus

$$\int_\Omega \nabla (e^u - 1) \cdot \nabla \left(\frac{\varphi}{e^{T_k(u)}} \right) dx = \int_\Omega \nabla (w + h) \cdot \nabla \left(\frac{\varphi}{e^{T_k(u)}} \right) dx = \int_\Omega e^{-T_k(u)} \varphi \, d\mu.$$

This implies

$$\int_\Omega |\nabla u|^2 \varphi \, dx = \int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \nabla (e^u - 1) \cdot \frac{\nabla \varphi}{e^u} \, dx$$
$$= \lim_{k \to \infty} \int_\Omega \nabla (e^u - 1) \cdot \frac{\nabla \varphi}{e^{T_k(u)}} \, dx$$
$$= \lim_{k \to \infty} \left(\int_\Omega \nabla (e^u - 1) \cdot \nabla \left(\frac{\varphi}{e^{T_k(u)}} \right) dx + \int_\Omega \nabla (e^u - 1) \cdot \nabla T_k(u) \frac{\varphi}{e^{T_k(u)}} \, dx \right)$$

$$(8)\qquad = \lim_{k\to\infty}\left(\int_\Omega \nabla(e^u-1)\cdot\nabla\left(\frac{\varphi}{e^{T_k(u)}}\right)dx + \int_\Omega \nabla e^{T_k(u)}\cdot\nabla T_k(u)\frac{\varphi}{e^{T_k(u)}}\,dx\right)$$
$$= \lim_{k\to\infty}\left(\int_\Omega e^{-T_k(u)}\varphi\,d\mu + \int_\Omega \varphi\,\nabla T_k(u)\cdot\nabla T_k(u)\,dx\right)$$
$$= \lim_{k\to\infty}\left(\int_\Omega e^{-T_k(u)}\varphi\,d\mu + \int_\Omega |\nabla T_k(u)|^2\varphi\,dx\right)$$
$$= \int_\Omega e^{-u}\varphi\,d\mu + \int_\Omega |\nabla u|^2\varphi\,dx,$$

and hence

$$\int_\Omega e^{-u}\varphi\,d\mu = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. We obtain that $\mu(\{u<\infty\})=0$. Since u is superharmonic, we have $\mathrm{cap}_2(\{u=\infty\})=0$. Thus $\mu \perp \mathrm{cap}_2$.

Next prove the converse. Clearly $u \in W_0^{1,2}(\Omega)$. Function e^u-1 is superharmonic and by the Riesz decomposition theorem

$$e^{u(x)}-1 = \int_\Omega G(x,y)d\mu(y) + h(x),$$

where h is harmonic. Since h cannot take values $\pm\infty$ in Ω, we have $e^{u(x)}=\infty$ if and only if $v_{\mu,\Omega}(x) = \int_\Omega G(x,y)d\mu(y) = \infty$. By lemma 3.2 we have $\mu(\{v_{\mu,\Omega}(x)<\infty\})=0$. This implies

$$(9)\qquad \mu(\{e^{u(x)}<\infty\}) = 0.$$

Let $k>0$, $\varphi\in C^\infty(\Omega)$ and $D \subset\subset \Omega$ smooth such that $\mathrm{spt}(\varphi)\subset\subset D$. As in the proof of the first part, we find a renormalized solution w in D and by lemma 2.1 $w = v_{\mu,D}$. So we have $e^u - 1 = w + h$ in D. Now, $e^{-T_k(u)}\varphi$ is a valid test function for $w+h$ and we find, calculating as earlier in (8), that

$$\int_0 \nabla u\cdot\nabla\varphi\,dx = \int_\Omega \nabla e^u\cdot\frac{\nabla\varphi}{e^u}\,dx = \int_\Omega e^{-u}\varphi\,d\mu + \int_\Omega |\nabla u|^2\varphi\,dx = \int_\Omega |\nabla u|^2\varphi\,dx,$$

since by equation (9) we have $\int_\Omega e^{-u}\varphi\,d\mu = 0$. □

References

[1] Abdellaoui, B., A. Dall'Aglio, and I. Peral: Some remarks on elliptic problems with critical growth in the gradient. - J. Differential Equations 222:1, 2006, 21–62.

[2] Abdellaoui, B., A. Dall'Aglio, and I. Peral: Corrigendum to "Some remarks on elliptic problems with critical growth in the gradient" - J. Differential Equations 246:7, 2009, 2988–2990.

[3] Adams, D. R., and L. I. Hedberg: Function spaces and potential theory. Springer-Verlag, Berlin–Heidelberg, 1996.

[4] Ancona, A.: Elliptic operators, conormal derivatives and positive parts of functions. With an appendix by Haïm Brezis. - J. Funct. Anal. 257, 2009, 2124–2158.

[5] Armitage, D. H., and S. J. Gardiner: Classical potential theory. - Springer-Verlag, London, 2001.

[6] Barles, G., A.-P. Blanc, C. Georgelin, and M. Kobylanski: Remarks on the maximum principle for nonlinear elliptic PDEs with quadratic growth conditions. - Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28:3, 1999, 381–404.

[7] Boccardo, L., F. Murat, and J.-P. Puel: Existence de solutions non bornées pour certaines equations quasi-linéaires. - Portugal. Math. 41:1-4, 1982, 507–534.

[8] Boccardo, L., S. Segura de León, and C. Trombetti: Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term. - J. Math. Pures Appl. (9) 80:9, 2001, 919–940.

[9] Brezis, H., and A. C. Ponce: Kato's inequality when Δu is a measure. - C. R. Math. Acad. Sci. Paris 338:8, 2004, 599–604.

[10] Dal Maso, G., F. Murat, L. Orsina, and A. Prignet: Renormalized solutions of elliptic equations with general measure data. - Ann. Scuola Norm. Sup. Pisa Cl. Sci. 28, 1999, 741–808.

[11] Ferone, V., and F. Murat: Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small. - Nonlinear Anal. 42, 2000, 1309–1326.

[12] Fukushima, M., K. Sato, and S. Taniguchi: On the closable parts of pre-Dirichlet forms and the fine supports of underlying measures. - Osaka J. Math. 28:3, 1991, 517–535.

[13] Hajłasz, P.: A counterexample to the L^p-Hodge decomposition. - Banach Center Publ. 33, 1996, 79–83.

[14] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear potential theory of degenerate elliptic equations. - Dover Publications, 2006.

[15] Jin, T., V. Maz'ya, and J. Van Schaftingen: Pathological solutions to elliptic problems in divergence form with continuous coefficients. - C. R. Math. Acad. Sci. Paris 347:13-14, 2009, 773–778.

[16] Kazdan, J. L., and R. J. Kramer: Invariant criteria for existence of solutions to second-order quasi-linear elliptic equations. - Comm. Pure Math. 31:5, 1978, 619–645.

Received 29 January 2010