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AN N-DIMENSIONAL VERSION OF THE BEURLING–AHLFORS EXTENSION

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Abstract. We extend delta-monotone quasiconformal mappings from dimension n to $n+1$ while preserving both monotonicity and quasiconformality. This gives an analytic proof of the extendability of quasiconformal mappings that can be factored into bi-Lipschitz and delta-monotone mappings. In the case $n=1$ our approach yields a refinement of the Beurling–Ahlfors extension.

1. Introduction

Extension problem. Given a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ of class $\mathscr{A}$, find $F\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ of class $\mathscr{A}$ such that the restriction of F to $\mathbf{R}^n$ agrees with f.

Let us introduce coordinate notation $x=(x^1,\dots,x^n)$ and $f=(f^1,\dots,f^n)$. By setting $F^i=f^i$ for $i=1,\dots,n$ and $F^{n+1}=x^{n+1}$ one immediately obtains a solution to the extension problem for many classes $\mathscr{A}$ such as continuous ($\mathscr{A}=C^0$), smooth ($\mathscr{A}=C^k$), homeomorphic, diffeomorphic, and (bi-)Lipschitz mappings.

When $\mathscr{A}=\mathscr{QC}$, the class of quasiconformal mappings, the extension problem is much more difficult. It was solved

- for $n=1$ by Beurling and Ahlfors [4] in 1956,
- for $n=2$ by Ahlfors [1] in 1964,
- for $n\le 3$ by Carleson [8] in 1974, and
- for all $n\ge 1$ by Tukia and Väisälä [16] in 1982.

The Tukia–Väisälä extension uses, among other things, Sullivan's theory [15] of deformations of Lipschitz embeddings. Except for the low dimensional case $n\le 2$, no analytic solution of the extension problem is available. Our goal is to give an explicit extension for the subgroup of $\mathscr{QC}$ generated by bi-Lipschitz and delta-monotone mappings. Quasiconformal mappings can be defined as orientation-preserving quasisymmetric mappings [11, 17].

Definition 1.1. A homeomorphism $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is quasisymmetric if there is a homeomorphism $\eta\colon [0,\infty)\to[0,\infty)$ such that

$$\frac{|f(x)-f(z)|}{|f(y)-f(z)|} \le \eta\left(\frac{|x-z|}{|y-z|}\right) \tag{1.1}$$

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for $x, y, z \in \mathbf{R}^n$, $z \neq y$.

One can say that quasisymmetry is a three-point condition. But there are two subclasses of $\mathscr{QC}$ that are defined by *two-point* conditions, namely bi-Lipschitz class $\mathscr{BL}$ and the class of nonconstant delta-monotone mappings [2, Chapter 3]. Recall that a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is *monotone* if

$$\langle f(x) - f(y), x - y\rangle \geq 0 \quad \text{for all } x, y \in \mathbf{R}^n. \tag{1.2}$$

We called f *delta-monotone* if there exists $\delta > 0$ such that

$$\langle f(x) - f(y), x - y\rangle \geq \delta |f(x) - f(y)||x - y| \quad \text{for all } x, y \in \mathbf{R}^n. \tag{1.3}$$

The class of nonconstant delta-monotone mappings is denoted by $\mathscr{DM}$. When we want to specify the value of δ we write that f is δ-monotone.

In contrast to the bi-Lipschitz case, the extension problem for the class $\mathscr{DM}$ cannot be solved by means of the trivial extension. For example, the mapping $f(x) = |x|^p x$, $p > -1$, belongs to $\mathscr{DM}$ but its trivial extension does not (unless $p = 0$).

Main result. *Let $n \geq 2$. For any mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ of class $\mathscr{DM}$ there exists $F\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ of class $\mathscr{DM}$ such that the restriction of F to $\mathbf{R}^n$ agrees with f.*

Let $\mathscr{QC}_d \subset \mathscr{QC}$ be the group generated by $\mathscr{BL}$ and $\mathscr{DM}$ under composition. In other words, f belongs to $\mathscr{QC}_d$ if it can be factored into bi-Lipschitz and delta-monotone mappings. This should be compared with the notion of polar factorization of mappings introduced by Brenier [6]. By combining our main result with the trivial extension of bi-Lipschitz mappings, we obtain a solution to the extension problem for $\mathscr{QC}_d$.

Corollary 1.2. Let $n \geq 2$. For any mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ of class $\mathscr{QC}_d$ there exists $F\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ of class $\mathscr{QC}_d$ such that the restriction of F to $\mathbf{R}^n$ agrees with f.

Let us now describe the extension process, which can be viewed as an n-dimensional version of the Beurling–Ahlfors extension [4]. Suppose $f \in \mathscr{DM}$. Let $\mathbf{R}^{n+1}_+ = \mathbf{R}^n \times [0, \infty)$ and

$$\phi(x) = (2\pi)^{-\frac{n}{2}} e^{-|x|^2/2}, \quad x \in \mathbf{R}^n. \tag{1.4}$$

We define $F\colon \mathbf{R}^{n+1}_+ \to \mathbf{R}^{n+1}_+$ by

$$F^i(x, t) = \int_{\mathbf{R}^n} f^i(x + ty)\phi(y)\, dy, \quad i = 1, \dots, n, \tag{1.5}$$

$$F^{n+1}(x, t) = \int_{\mathbf{R}^n} \langle f(x + ty), y\rangle \phi(y)\, dy \tag{1.6}$$

where $x \in \mathbf{R}^n$, $t \geq 0$ (see §4 for the convergence of these integrals). Observe that $F(x, 0) = (f(x), 0)$. Furthermore, $F^{n+1}(x, t) \geq 0$ because

$$\int_{\mathbf{R}^n} \langle f(x + ty), y\rangle \phi(y)\, dy = \int_{\mathbf{R}^n} \langle f(x + ty) - f(x), y\rangle \phi(y)\, dy \geq 0$$

due to the monotonicity of f. Finally, we extend F to $\mathbf{R}^{n+1}$ by reflection

$$F^i(x, t) = F^i(x, -t), \quad i = 1, \dots, n, \quad \text{and} \quad F^{n+1}(x, t) = -F^{n+1}(x, -t).$$

Theorem 1.3. *Let $n \geq 2$. If $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is δ-monotone, then $F\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is δ_1-monotone where δ_1 depends only on δ and n. In addition, $F\colon \mathbf{H}^{n+1} \to \mathbf{H}^{n+1}$ is bi-Lipschitz in the hyperbolic metric.*

Here $\mathbf{H}^{n+1} = \mathbf{R}^n \times (0,\infty)$ and the hyperbolic metric on $\mathbf{H}^{n+1}$ is $|dx|/x^{n+1}$. Theorem 1.3 can be also formulated for $n = 1$, in which case it becomes a refinement of the Beurling–Ahlfors extension theorem.

Proposition 1.4. *If $f\colon \mathbf{R} \to \mathbf{R}$ is increasing and quasisymmetric, then $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ is δ_1-monotone where δ_1 depends only on η in Definition 1.1. Furthermore, $F\colon \mathbf{H}^2 \to \mathbf{H}^2$ is bi-Lipschitz in the hyperbolic metric.*

Fefferman, Kenig and Pipher [9, Lemma 4.4] proved that F in Proposition 1.4 is quasiconformal. Proposition 1.4 was originally proved in [12] using their result. In this paper we give a direct proof.

Theorem 1.3 has an application to mappings with a convex potential [7], i.e., those of the form $f = \nabla u$ with u convex. The basic properties and examples of quasiconformal mappings with a convex potential are given in [13].

Corollary 1.5. Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^n$, $n \geq 2$, is a K-quasiconformal mapping with a convex potential. Then f can be extended to a K_1-quasiconformal mapping $F\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ with a convex potential, where K_1 depends only on K and n.

2. Preliminaries

Let $e_1, \dots, e_{n+1}$ be the standard basis of $\mathbf{R}^{n+1}$. All vectors are treated as column vectors. The transpose of a vector v is denoted by v^T. We use the operator norm $\|\cdot\|$ for matrices. A Borel measure μ on $\mathbf{R}^n$ is *doubling* if there exists $\mathscr{D}_\mu$, called the doubling constant of μ, such that

$$\mu(2B) \leq \mathscr{D}_\mu \mu(B)$$

for all balls $B = B(x,r)$. Here $2B = B(x,2r)$.

The geometric definition of class $\mathscr{QC}$ given in the introduction is equivalent to the following analytic definition [11, 17].

Definition 2.1. A homeomorphism $f\colon \mathbf{R}^n \to \mathbf{R}^n$ ($n \geq 2$) is quasiconformal if $f \in W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^n)$ and there exists a constant K such that the differential matrix $Df(x)$ satisfies the distortion inequality

$$\|Df(x)\|^n \leq K \det Df(x) \quad \text{a.e. in } \mathbf{R}^n.$$

Delta-monotone mappings also have an analytic definition.

Lemma 2.2. *Let Ω be a convex domain in $\mathbf{R}^n$, $n \geq 2$. Suppose $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ is continuous. The following are equivalent:*

(i) f is δ-monotone in Ω for some $\delta > 0$; that is, (1.3) holds for all $x, y \in \Omega$;

(ii) there exists $\delta > 0$ such that for a.e. $x \in \Omega$ the matrix $Df(x)$ satisfies

$$v^T Df(x) v \geq \delta |Df(x)v||v| \quad \text{for every vector } v \in \mathbf{R}^n;$$

(iii) there exists $\gamma > 0$ such that for a.e. $x \in \Omega$ the matrix $Df(x)$ satisfies

$$v^T Df(x) v \geq \gamma \|Df(x)\| |v|^2 \quad \text{for every vector } v \in \mathbf{R}^n.$$

The constants δ and γ depend only on each other.

Proof. The equivalence of (i) and (ii), with the same constant δ, was proved in [12, p. 397]. It is obvious that (iii) implies (ii) with $\delta = \gamma$. It remains to establish the converse implication (ii) $\Longrightarrow$ (iii). To this end we need the following

Claim: If a real square matrix A satisfies

$$v^T Av \geq \delta |Av||v| \quad \text{for every } v \in \mathbf{R}^n,$$

then

$$|Av| \geq c\|A\||v| \quad c = c(\delta) > 0. \tag{2.1}$$

Although this claim is known, even with a sharp constant [3], we give a proof for the sake of completeness. It suffices to estimate $|Av|$ from below under the assumptions that $Av \neq 0$ and $\|A\| = 1 = |v|$. Let u be a unit vector in $\mathbf{R}^n$ such that $|Au| = 1$. Replacing u by $-u$ if necessary we may assume that $u^T Av + v^T Au \leq 0$. Let $\lambda = \sqrt{|Av|}$. On one hand we have

$$(\lambda u + v)^T A(\lambda u + v) \leq \lambda^2 u^T Au + v^T Av \leq \lambda^2 + \lambda^2 = 2\lambda^2. \tag{2.2}$$

On the other hand

$$(\lambda u + v)^T A(\lambda u + v) \geq \delta|\lambda Au + Av||\lambda u + v| \geq \delta(\lambda - \lambda^2)(1 - \lambda). \tag{2.3}$$

Combining (2.2) and (2.3) we obtain $2\lambda \geq \delta(1 - \lambda)^2$, hence

$$\lambda \geq \delta^{-1} + 1 - \sqrt{(\delta^{-1} + 1)^2 - 1} > 0.$$

This proves the claim. □

3. Delta-monotone mappings and doubling measures

The following result shows that $\mathscr{DM} \subset \mathscr{QC}$. In particular, $f \in \mathscr{DM}$ implies that f is a continuous Sobolev mapping, and therefore (ii)–(iii) of Lemma 2.2 hold.

Proposition 3.1. [12, Theorem 6] *Every nonconstant δ-monotone mapping is η-quasisymmetric where η depends only on δ.*

It is well-known that quasisymmetric mappings are closely related to doubling measures [11]. The following lemma is another instance of this relation.

Lemma 3.2. *For any nonconstant δ-monotone mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ $(n \geq 2)$ the measure $\mu = \|Df(x)\|\,dx$ is doubling. The doubling constant $\mathscr{D}_\mu$ depends only on δ and n.*

Proof. Recall that f is quasisymmetric. Lemma 3.2 in [14] implies the existence of a constant $C = C(\delta, n)$ such that

$$C^{-1}\frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \leq \frac{1}{|B|}\int_B \|Df\|\,dx \leq C\frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \tag{3.1}$$

for all balls $B \subset \mathbf{R}^n$. Since $\operatorname{diam} f(2B) \leq C \operatorname{diam} f(B)$ with $C = C(\eta)$, the lemma follows. □

Recall that $\phi\colon \mathbf{R}^n \to (0, \infty)$ is the Gaussian kernel (1.4). Let $\mathbf{B} = B(0, 1)$ be the open unit ball in $\mathbf{R}^n$.

Lemma 3.3. *Let μ be a doubling measure in $\mathbf{R}^n$ and $p \geq 0$. Let Ω be either $\mathbf{R}^n$ or the half space $\{y\colon \langle y, \xi\rangle \geq 0\}$ for some $\xi \in \mathbf{R}^n$. Then*

$$C^{-1}\mu(\mathbf{B}) \leq \int_\Omega |y|^p \phi(y)\, d\mu(y) \leq C\mu(\mathbf{B}) \tag{3.2}$$

where the constant C depends only on $\mathscr{D}_\mu$, p and n.

Proof. We begin by estimating the integral in (3.2) from above as follows

$$\int_{\mathbf{R}^n} |y|^p \phi(y)\, d\mu(y) = \int_{\mathbf{B}} |y|^p \phi(y)\, d\mu(y) + \sum_{k=0}^{\infty} \int_{2^k<|y|\leq 2^{k+1}} |y|^p \phi(y)\, d\mu(y),$$

where

$$\int_{\mathbf{B}} |y|^p \phi(y)\, d\mu(y) \leq \phi(0)\mu(\mathbf{B}) = (2\pi)^{-\frac{n}{2}}\mu(\mathbf{B})$$

and

$$\begin{aligned}\int_{2^k<|y|\leq 2^{k+1}} |y|^p \phi(y)\, d\mu(y) &\leq 2^{p(k+1)}(2\pi)^{-\frac{n}{2}} e^{-2^{2k-1}} \mu(B(0,2^{k+1}))\\ &\leq 2^{p(k+1)}(2\pi)^{-\frac{n}{2}} e^{-2^{2k-1}} \mathscr{D}_\mu^{k+1}\mu(\mathbf{B}).\end{aligned}$$

Summing over $k = 0, 1, 2\dots$ we obtain

$$\int_{\mathbf{R}^n} \phi(y)\, d\mu(y) \leq C\mu(\mathbf{B})$$

where $C = C(\mathscr{D}_\mu, p, n) > 0$.

We turn to the left side of (3.2). The inequality

$$|y|^p\phi(y) \geq \frac{e^{-1/2}}{2^p(2\pi)^{n/2}} \quad \text{for } \frac{1}{2} \leq |y| \leq 1$$

implies

$$\int_\Omega |y|^p \phi(y)\, d\mu(y) \geq \frac{e^{-1/2}}{2^p(2\pi)^{n/2}}\mu(\Omega \cap \{1/2 \leq |y| \leq 1\}).$$

Since $\mu(\Omega \cap \{1/2 \leq |y| \leq 1\}) \geq \mathscr{D}_\mu^{-3}\mu(\mathbf{B})$, the left side of (3.2) follows. □

4. Proof of main results

Proof of Theorem 1.3. Since f is quasisymmetric by Proposition 3.1, it satisfies the growth condition $|f(x)| \leq \alpha|x|^p + \beta$ for some constants α, β, p, see [11, Theorem 11.3]. Therefore, the integrals (1.5) and (1.6) converge and F is C^∞-smooth in $\mathbf{H}^{n+1}$. Let $\gamma = \gamma(\delta) > 0$ be as in part (iii) of Lemma 2.2.

Our first step is to prove that for $(x,t) \in \mathbf{H}^{n+1}$ the matrix $\mathscr{B} := DF(x,t)$ satisfies the condition

$$w^T\mathscr{B}w \geq \gamma_1\|\mathscr{B}\||w|^2 \qquad \text{for every vector } w \in \mathbf{R}^{n+1} \tag{4.1}$$

where $\gamma_1 = \gamma_1(\delta, n) > 0$. Fix $x \in \mathbf{R}^n$ and $t > 0$. We compute the partial derivatives of F at $(x,t) \in \mathbf{H}^{n+1}$ as follows:

$$\begin{aligned}
\frac{\partial F^i}{\partial x_j} &= \int_{\mathbf{R}^n} f^i_j(x+ty)\phi(y)\,dy, \quad 1 \le i,j \le n;\\
\frac{\partial F^i}{\partial t} &= \int_{\mathbf{R}^n} \sum_{j=1}^n f^i_j(x+ty)y^i\phi(y)\,dy, \quad 1 \le i \le n;\\
\frac{\partial F^{n+1}}{\partial x_j} &= \int_{\mathbf{R}^n} \sum_{i=1}^n f^i_j(x+ty)y^j\phi(y)\,dy, \quad 1 \le j \le n;\\
\frac{\partial F^{n+1}}{\partial t} &= \int_{\mathbf{R}^n} \sum_{i=1}^n \sum_{j=1}^n f^i_j(x+ty)y^iy^j\phi(y)\,dy.
\end{aligned}$$

To simplify formulas we write $A(y) = Df(x+ty)$ and let $B(y)$ be the $(n+1)\times(n+1)$ matrix written in block form below,

$$B(y) = \left(\begin{array}{c|c} A(y) & A(y)y \\ \hline y^TA(y) & y^TA(y)y \end{array}\right). \tag{4.2}$$

With this notation we have

$$DF(x,t) = \int_{\mathbf{R}^n} B(y)\phi(y)\,dy. \tag{4.3}$$

First we show that the norm of $\mathscr{B}$ is dominated by the quantity

$$\alpha := \int_{B(0,1)} \|A(y)\|\,dy.$$

Indeed,

$$\|\mathscr{B}\| \le \int_{\mathbf{R}^n} \|B(y)\|\phi(y)\,dy \le \int_{\mathbf{R}^n} \|A(y)\|(1+|y|)^2\phi(y)\,dy.$$

By Lemma 3.2 the measure $\mu = \|A(y)\|\,dy$ is doubling. Applying Lemma 3.3 we obtain

$$\|\mathscr{B}\| \le C\alpha, \quad C = C(\delta, n). \tag{4.4}$$

Next we estimate the quadratic form $w \mapsto w^T\mathscr{B}w$ generated by $\mathscr{B}$ from below. For this we fix a vector $w \in \mathbf{R}^{n+1}$, written as $w = v + se_{n+1}$ with $v \in \mathbf{R}^n$ and $s \in \mathbf{R}$. It is easy to see that

$$w^TB(y)w = (v+sy)^TA(y)(v+sy).$$

Let $\Omega = \{y \in \mathbf{R}^n \colon \langle v, sy\rangle \ge 0\}$. Then

$$\begin{aligned}
w^T\mathscr{B}w &= \int_{\mathbf{R}^n} \left\{(v+sy)^TA(y)(v+sy)\right\}\phi(y)\,dy\\
&\ge \gamma \int_{\mathbf{R}^n} \|A(y)\||v+sy|^2\phi(y)\,dy\\
&\ge \gamma \int_{\Omega} \|A(y)\||v+sy|^2\phi(y)\,dy\\
&\ge \gamma|v|^2 \int_{\Omega} \|A(y)\|\phi(y)\,dy + \gamma s^2 \int_{\Omega} \|A(y)\||y|^2\phi(y)\,dy.
\end{aligned}$$

Applying Lemma 3.3 with $\mu = \|A(y)\|\,dy$ we obtain

$$w^T \mathscr{B} w \geq c\alpha\gamma(|v|^2 + s^2) = c\alpha\gamma|w|^2, \quad c = c(\delta, n). \tag{4.5}$$

Combining (4.4) and (4.5) we obtain (4.1) with $\gamma_1 = (c/C)\gamma$. By virtue of Lemma 2.2 F is δ_1-monotone in the upper half-space $\mathbf{H}^{n+1}$ where $\delta_1 = \delta_1(\delta, n)$. By symmetry, F is also δ_1-monotone in the lower half-space.

To prove that F is δ_1-monotone in the entire space $\mathbf{R}^{n+1}$, we consider two points $a, b \in \mathbf{R}^{n+1}$ such that the line segment $[a, b]$ crosses the hyperplane $\mathbf{R}^n$ at some point c. We have

$$\begin{aligned}\langle F(a) - F(b), a - b\rangle &= \langle f(a) - f(c), a - b\rangle + \langle F(c) - F(b), a - b\rangle \\ &\geq \delta_1|F(a) - F(c)||a - b| + \delta_1|F(c) - F(b)||a - b| \\ &\geq \delta_1|F(a) - F(b)||a - b|.\end{aligned}$$

Therefore, $F \in \mathscr{DM}$.

It remains to show that $F\colon \mathbf{H}^{n+1} \to \mathbf{H}^{n+1}$ is bi-Lipschitz in the hyperbolic metric. Since $F \in \mathscr{QC}$ and $\mathbf{H}^{n+1}$ is a geodesic space, it suffices to prove that

$$\|DF(x, t)\| \approx \frac{F^{n+1}(x, t)}{t}. \tag{4.6}$$

Here $X \approx Y$ means that X and Y are comparable, i.e., $C^{-1}Y \leq X \leq CY$ where $C = C(\delta, n)$. It follows from (4.4) and (4.5) that $\|DF(x, t)\|$ is comparable to the integral average of $\|Df\|$ over the ball $B(x, t)$. By (3.1) this average is comparable to $t^{-1} \operatorname{diam} f(B(x, t))$. The quasisymmetry of F implies (cf. [11, 11.18])

$$\operatorname{diam} f(B(x, t)) \approx |F(x, t) - F(x, t/2)| \approx F^{n+1}(x, t).$$

This proves (4.6). □

Proof of Proposition 1.4. The proof of Theorem 1.3 also works in the case $n = 1$ with the following interpretation. Since quasisymmetric mappings on the line need not be absolutely continuous [4], the derivative f' must be understood in the sense of distributions. In fact, $\mu := f'$ is a positive doubling measure with $\mathscr{D}_\mu = \mathscr{D}_\mu(\eta)$ [11, 13.20]. Lemma 3.2 is not needed in this case. The rest of the proof carries over with $\gamma = 1$ and $\gamma_1 = \gamma_1(\mathscr{D}_\mu)$. □

Proof of Corollary 1.5. According to [12, Lemma 18], a K-quasiconformal mapping with a convex potential is also δ-monotone with $\delta = \delta(K, n)$. Let F be the δ_1-monotone extension of f provided by Theorem 1.3. Since the differential matrix Df is symmetric, the formulas (4.2) and (4.3) show that DF is symmetric as well. In addition, DF is positive semidefinite by Lemma 2.2. Thus, $F = \nabla U$ for some convex function $U\colon \mathbf{R}^{n+1} \to \mathbf{R}$. □

5. Concluding remarks

Corollary 1.2 naturally leads one to wonder how large the group $\mathscr{QC}_d$ really is. It is easy to give an example of a planar quasiconformal mapping that cannot be factored into delta-monotone mappings. Indeed, the logarithmic spiral map

$$S(\rho e^{i\theta}) = \rho e^{i(\theta + \log \rho)} \tag{5.1}$$

has this property, which can be verified as follows. Let us say that a simple curve $\Gamma\colon [0, 1] \to \mathbf{C}$ is of *bounded twist* if for any $a \in [0, 1]$ there is a bounded continuous branch of the function $t \mapsto \arg(\Gamma(t) - \Gamma(a))$, $t \in [0,1] \setminus \{a\}$. For any delta-monotone mapping $F\colon \mathbf{C} \to \mathbf{C}$ the principal value of

$$\arg \frac{F(z) - F(w)}{z - w}, \quad z, w \in \mathbf{C},\ z \neq w,$$

defines a bounded continuous function on $\mathbf{C}^2 \setminus \{z = w\}$. It follows that the image of a curve of bounded twist is invariant under delta-monotone mappings. But the image of the line segment $[0,1]$ under the mapping S in (5.1) is not of bounded twist.

It is not as easy to give an example of an element in $\mathscr{QC} \setminus \mathscr{QC}_d$. For instance, the mapping S belongs to $\mathscr{QC}_d$ by virtue of being bi-Lipschitz. This motivates the following question:

Question 5.1. What are the obstructions for factorization of a quasiconformal mapping into bi-Lipschitz and delta-monotone factors?

Inspired by the example (5.1), one could expect to find such an obstruction in the form of a $\mathscr{QC}_d$-invariant property of curves. For instance, both bi-Lipschitz and delta-monotone mappings take smooth curves into rectifiable curves [2, Theorem 3.11.7]. However, this is no longer true of their compositions; in fact, the image of a line segment under a planar $\mathscr{QC}_d$ mapping can have Hausdorff dimension arbitrarily close to 2. We briefly sketch this construction,leaving details to the reader. The first step is to find a bi-Lipschitz mapping $g\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that $g(\mathbf{R})$ contains a planar Cantor set E of dimension $0 < \beta < 1$ (see Lemma 3.1 [5] and the comment after its proof). Then one constructs a delta-monotone mapping $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that the Hausdorff dimension of $h(E)$ is equal to α, for a given $\alpha \in (0,2)$ (see the proof of [10, Theorem 5]). Finally, let $f = h \circ g$.

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