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A CRITERION OF NORMALITY BASED ON A SINGLE HOLOMORPHIC FUNCTION II

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Abstract. In this paper, we continue to discuss normality based on a single holomorphic function. We obtain the following result. Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbf{C}$. Let $k \geq 2$ be an integer and let $h\,(\not\equiv 0)$ be a holomorphic function on D, such that $h(z)$ has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$: (a) $f(z) = 0 \Longrightarrow f'(z) = h(z)$, and (b) $f'(z) = h(z) \Longrightarrow |f^{(k)}(z)| \leq c$, where c is a constant. Then $\mathcal{F}$ is normal on D. A geometrical approach is used to arrive at the result that significantly improves a previous result of the authors which had already improved a result of Chang, Fang and Zalcman. We also deal with two other similar criterions of normality. Our results are shown to be sharp.

1. Introduction

In [11], Pang and Zalcman proved the following theorem.

Theorem PZ. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subset \mathbf{C}$, all of whose zeros have multiplicity at least k, where $k \geq 1$ is an integer. Suppose there exist constants $b \neq 0$ and $h > 0$ such that, for every $f \in \mathcal{F}$, $f(z) = 0 \Longleftrightarrow f^{(k)}(z) = b$ and $f(z) = 0 \Longrightarrow 0 < |f^{(k+1)}(z)| \leq h$. Then $\mathcal{F}$ is a normal family on D.*

Then, in [1], Chang, Fang and Zalcman proved the following result.

Theorem CFZ1. [1, Theorem 4] *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbf{C}$. Let $k \geq 2$ be an integer, and let $h(z) \neq 0$ be a function analytic in D. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Longrightarrow f'(z) = h(z)$, *and*
(b) $f'(z) = h(z) \Longrightarrow |f^{(k)}(z)| \leq c$, *where c is a constant.*

Then $\mathcal{F}$ is normal on D.

And in [4], we replaced the condition $h(z) \neq 0$ with $h(z) \not\equiv 0$ and obtained the following result.

Theorem LN. *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbf{C}$. Let $k \geq 2$ be an integer, and let $h(z)\,(\not\equiv 0)$ be a holomorphic function on D, all of*

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whose zeros have multiplicity at most $k-1$, that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:

(a) $f(z) = 0 \Longrightarrow f'(z) = h(z)$, *and*
(b) $f'(z) = h(z) \Longrightarrow |f^{(k)}(z)| \le c$, *where c is a constant.*

Then $\mathcal{F}$ is normal on D.

We now pose the following question: Can the restriction for the zeros of $h(z)$ with multiplicity at most $k-1$ be dropped? In this paper, we continue to study the above problem and obtain an affirmative answer.

Theorem 1. *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbf{C}$. Let $k \ge 2$ be an integer, and let $h(z)\,(\not\equiv 0)$ be a holomorphic function on D that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Longrightarrow f'(z) = h(z)$, *and*
(b) $f'(z) = h(z) \Longrightarrow |f^{(k)}(z)| \le c$, *where c is a constant.*

Then $\mathcal{F}$ is normal on D.

Also in [1], the case for the kth derivative was considered and the following result was proved.

Theorem CFZ2. [1, Theorem 1] *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbf{C}$, all of whose zeros have multiplicity at least k, where $k \ne 2$ is a positive integer, and let $h(z) \ne 0$ be a function analytic in D. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Longrightarrow f^{(k)}(z) = h(z)$, *and*
(b) $f^{(k)}(z) = h(z) \Longrightarrow |f^{(k+1)}(z)| \le c$, *where c is a constant.*

Then $\mathcal{F}$ is normal on D.

For the case $k = 2$, the following result was obtained.

Theorem CFZ3. [1, Theorem 3] *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbf{C}$, all of whose zeros are multiple, where $s \ge 4$ is an even integer; and let $h(z) \ne 0$ be a function analytic in D. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Longrightarrow f''(z) = h(z)$, *and*
(b) $f''(z) = h(z) \Longrightarrow |f'''(z)| + |f^{(s)}(z)| \le c$, *where c is a constant.*

Then $\mathcal{F}$ is normal on D.

In view of the improvement of Theorems CFZ1 and LN via Theorem 1, the question that naturally arises concerning Theorems CFZ2 and CFZ3 is whether the condition $h(z) \ne 0$, $z \in D$ can be weakened to "$h \not\equiv 0$". It turns out that the answer is negative in both cases. It is negative even if h has no common zero with any $f \in \mathcal{F}$ (like in Theorem 1). To construct the first example, concerning Theorem CFZ2, we first need to mention the following famous result of Lucas.

Theorem Lu. [5], [6, p. 22] *Let $P(z)$ be a nonconstant polynomial. Then all the zeros of $P'(z)$ lie in the convex hull H of the zeros of $P(z)$. Moreover, there are no zeros of $P'(z)$ on the boundary of H, unless this zero is a multiple zero of $P(z)$ or the zeros of $P(z)$ are colinear.*

Example 1. Let $r \geq 1$ and $k \geq 3$ be integers, $D = \Delta$ be the unit disc and $h(z) = z^r$. Define

$$f_n(z) = a_n \left(z^\ell - \frac{1}{n^\ell} \right)^k,$$

where $\ell = k + r$ and $a_n = \dfrac{n^{(k-1)\ell}}{k!\ell^k}$.

We have

$$f_n(z) = a_n \prod_{j=1}^{\ell} \left(z - \alpha_j^{(n)} \right)^k,$$

where $\alpha_j^{(n)} = \dfrac{\exp\left(i \dfrac{2\pi j}{\ell} \right)}{n}$, for $1 \leq j \leq \ell$. By calculation,

$$\begin{aligned} f_n^{(k)}\left(\alpha_j^{(n)}\right) &= k! a_n \prod_{t=1, t\neq j}^{\ell} \left(\alpha_j^{(n)} - \alpha_t^{(n)} \right)^k = k! a_n \left[\left(z^\ell - \frac{1}{n^\ell} \right)' \Bigg|_{z=\alpha_j^{(n)}} \right]^k \\ &= k! a_n \ell^k \left(\alpha_j^{(n)} \right)^{k(\ell-1)}. \end{aligned}$$

Thus,

$$\text{(1)} \qquad \arg\left[f_n^{(k)}\left(\alpha_j^{(n)} \right) \right] = (\ell - 1)k \cdot \frac{2\pi j}{\ell} = -\frac{2\pi k j}{\ell} = \frac{2\pi r i}{\ell} = \arg\left[z^r \Big|_{z=\alpha_j^{(n)}} \right].$$

Here the equalities are modulo 2π, and we used in the last equality that $r + k = \ell$.

We have

$$\text{(2)} \qquad \left| f_n^{(k)}\left(\alpha_j^{(n)} \right) \right| = \frac{k!\ell^k n^{\ell(k-1)}}{k!\ell^k} \left(\frac{1}{n} \right)^{k(\ell-1)} = \left(\frac{1}{n} \right)^r = |z^r| \Bigg|_{z=\alpha_j^{(n)}}.$$

From (1) and (2) we have that $f_n(z) = 0 \Longrightarrow f_n^{(k)}(z) = h(z)$, i.e., assumption (a) of Theorem CFZ2 holds.

In order to confirm (b) of Theorem CFZ2, set

$$\widetilde{f}_n(z) = f_n(z) - \frac{z^\ell}{\ell(\ell-1)\cdots(r+1)}.$$

We have $f_n^{(k)}(z) = h(z) \Longleftrightarrow \widetilde{f}_n^{(k)}(z) = 0$.

Now

$$\text{(3)} \qquad \widetilde{f}_n(z) = 0 \Longleftrightarrow \frac{n^{k(\ell-1)-r}}{k!\ell^k} \left(z^\ell - \frac{1}{n^\ell} \right)^k = \frac{z^\ell}{\ell(\ell-1)\cdots(r+1)}.$$

Suppose by negation that there exist a sequence $\{z_n\}_{n=1}^{\infty}$ ($z_n \to 0$) and a sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ ($k_n \underset{n\to\infty}{\longrightarrow} \infty$), such that $\widetilde{f}_{k_n}(z_n) = 0$. Then since $\dfrac{(k_n z_n)^\ell - 1}{(k_n z_n)^\ell} \underset{n\to\infty}{\longrightarrow} 1$, from (3), we get

$$\text{(4)} \qquad \frac{k_n^{(k-1)\ell} (k_n z_n)^{k\ell}}{k_n^{k\ell} z_n^\ell} \underset{n\to\infty}{\longrightarrow} \frac{k!\ell^k}{\ell(\ell-1)\cdots(r+1)}.$$

But the left hand side of (4) tends to ∞, as $n \to \infty$, a contradiction.

We deduce that there exists some $0 < C_1 < \infty$, such that every zero z_n of $\widetilde{f}_n$ satisfies $|z_n| \le \frac{C_1}{n}$. By Theorem Lu, we have also $|\widehat{z}_n| \le \frac{C_1}{n}$ for every $\widehat{z}_n$, which is a zero of $\widetilde{f}_n^{(k)}$. But those $\{\widehat{z}_n\}$ are exactly the points where $f_n^{(k)}(z) = h(z)$.

Hence $f_n^{(k)}(z) = h(z)$ implies that $|z| \le \frac{C_1}{n}$, and we have only to prove the following claim.

Claim 1. *There exists* $0 < C < \infty$, *such that* $|z| \le \frac{C_1}{n}$ *implies* $|f_n^{(k+1)}(z)| \le C$.

Proof. We have $f_n(z) = \frac{n^{(k-1)\ell}}{k!\ell^k}\left(z^\ell - \frac{1}{n^\ell}\right)^k = \frac{n^{(k-1)\ell}}{k!\ell^k}\sum_{j=0}^{k}\binom{k}{j}z^{\ell j}\left(\frac{1}{n}\right)^{\ell(k-j)}(-1)^{k-j}$. Thus, since $\ell j \ge k+1$ only for $j \ge 1$, we get that

$$f_n^{(k+1)}(z) = \frac{n^{(k-1)\ell}}{k!\ell^k}\sum_{j=1}^{k}\binom{k}{j}\left(\frac{1}{n}\right)^{\ell k-\ell j}(-1)^{k-j}\ell j(\ell j-1)\cdots(\ell j-k-1)z^{\ell j-k-1}.$$

Thus, if $|z| \le \frac{C_1}{n}$, then

$$\begin{aligned}|f_n^{(k+1)}(z)| &\le \frac{n^{(k-1)\ell}}{k!\ell^k}\sum_{j=1}^{k}\binom{k}{j}C_1^{\ell j-k-1}\ell j(\ell j-1)\cdots(\ell j-k-1)n^{k+1-\ell j}\cdot n^{\ell j-\ell k}\\ &= \frac{n^{k+1-\ell}}{k!\ell^k}\sum_{j=1}^{k}\binom{k}{j}C_1^{\ell j-k-1}\ell j(\ell j-1)\cdots(\ell j-k-1) \le C,\end{aligned}$$

where $C = \frac{1}{k!\ell^k}\sum_{j=1}^{k}\binom{k}{j}C_1^{\ell j-k-1}\ell j(\ell j-1)\cdots(\ell j-k-1)$. (Here we used that $k+1-\ell \le 0$.) Claim 1 is proved. □

Hence, $\{f_n\}$ with h satisfy (a) and (b) of Theorem CFZ2, but $\{f_n\}$ is not normal at $z = 0$.

Observe that when $k = 1$, then $a_n = \frac{1}{\ell} \not\to \infty$, and we do not get a non-normal family, as expected by Theorem 1.

The following example shows that the condition $h(z) \ne 0$ is essential also to Theorem CFZ3.

Example 2. (cf. [1, Ex. 4]) Let $s \ge 4$ be an even integer and consider the family $\mathcal{F} = \{f_n(z)\}_{n=1}^{\infty}$,

$$f_n(z) = \frac{n^s}{2s^2}\left(z^s - \frac{1}{n^s}\right)^2 \quad \text{on } \Delta.$$

Let $h(z) = z^{s-2}$. We have that

$$f_n(z) = \frac{n^s}{2s^2}\prod_{j=1}^{s}\left(z - \alpha_j^{(n)}\right)^2,$$

where $\alpha_j^{(n)} = \frac{\exp(i2\pi j/s)}{n}$, $1 \le j \le s$.

By calculation we have

$$f_n''(z) = \frac{n^s}{s}\left((2s-1)z^s - \frac{(s-1)}{n^s}\right)z^{s-2}, \tag{5}$$

$$\begin{aligned} f_n'''(z) &= \frac{n^s}{s}\left[(2s-1)(2s-2)z^s - \frac{(s-1)(s-2)}{n^s}\right]z^{s-3} \\ &= \frac{n^s}{s}(s-1)z^{s-3}\left[(4s-2)z^s - \frac{s-2}{n^s}\right], \end{aligned} \tag{6}$$

and

$$f_n^{(s)}(z) = \frac{n^s}{s}\left[(2s-1)(2s-2)\cdots(s+1)z^s - \frac{(s-1)!}{n^s}\right]. \tag{7}$$

Now, if $f_n(z) = 0$, then $z = \alpha_j^{(n)}$ for some $1 \le j \le s$, and thus $z^s = \frac{1}{n^s}$ and by (5), $f_n''(z) = z^{s-2} = h(z)$.

If $f_n''(z) = z^{s-2} = h(z)$, then by (5), $z = 0$ or $z = \alpha_j^{(n)}$, $1 \le j \le s$. By (6) and (7), we get

$$f_n^{(3)}(0) = 0, \quad f_n^{(s)}(0) = -\frac{(s-1)!}{n^s} \tag{8}$$

and

$$f_n^{(3)}\left(\alpha_j^{(n)}\right) = 3(s-1)\frac{1}{n^{s-3}}, \quad f_n^{(s)}\left(\alpha_j^{(n)}\right) = \frac{1}{s}\left[\frac{(2s-1)!}{s!} - (s-1)!\right]. \tag{9}$$

From (8) and (9), we see that the family $\mathcal{F}$ with h satisfy assumption (a) and (b) of Theorem CFZ3, but $\mathcal{F}$ is not normal at $z = 0$. Indeed, the reason must be that $h(0) = 0$.

In Example 1, we have that $f^{(k+1)}(z) \ne 0$ at the zero points of $f^{(k)}(z) - h(z)$. If we strengthen condition (b) of Theorem CFZ2 to be $f^{(k)}(z) = h(z) \Longrightarrow f^{(k+1)}(z) = 0$, then we can obtain the following normal criterion.

Theorem 2. *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbf{C}$, all of whose zeros have multiplicity at least k, where $k \ne 2$ is a positive integer. Let $h(z)\,(\not\equiv 0)$ be a holomorphic function on D, that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Longrightarrow f^{(k)}(z) = h(z)$, *and*
(b) $f^{(k)}(z) = h(z) \Longrightarrow f^{(k+1)}(z) = 0$.

Then $\mathcal{F}$ is normal on D.

Similarly, if we strengthen the condition (b) of Theorem CFZ3 to $f''(z) = h(z) \Longrightarrow f'''(z) = f^{(s)}(z) = 0$, then we can also obtain the normality criterion.

Theorem 3. *Let $\mathcal{F}$ be a family of functions holomorphic on a domain $D \subset \mathbf{C}$, all of whose zeros are multiple, where $s \ge 2$ is an even integer. Let $h(z)\,(\not\equiv 0)$ be a holomorphic function on D, that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Longrightarrow f''(z) = h(z)$, *and*
(b) $f''(z) = h(z) \Longrightarrow f'''(z) = f^{(s)}(z) = 0$.

Then $\mathcal{F}$ is normal on D.

Before we go to the proofs of the main results, let us set some notation. Throughout, D is a domain in $\mathbf{C}$. For $z_0 \in \mathbf{C}$ and $r > 0$, $\Delta(z_0, r) = \{z\colon |z - z_0| < r\}$ and $\Delta'(z_0, r) = \{z\colon 0 < |z - z_0| < r\}$. The unit disc will be denoted by Δ and $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. We write $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ on D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric, uniformly on compact subsets of D, and $f_n \Rightarrow f$ on D if the convergence is in the Euclidean metric. For a meromorphic function $f(z)$ in D and $a \in \widehat{\mathbf{C}}$, $\overline{E}_f(a) := \{z \in D\colon f(z) = a\}$. The spherical derivative of the meromorphic function f at the point z is denoted by $f^{\#}(z)$.

Frequently, given a sequence $\{f_n\}_1^\infty$ of functions, we need to extract an appropriate subsequence; and this necessity may recur within a single proof. To avoid the awkwardness of multiple indices, we again denote the extracted subsequence by $\{f_n\}$ (rather than, say, $\{f_{n_k}\}$) and designate this operation by writing "taking a subsequence and renumbering", or simply "renumbering". The same convention applies to sequences of constants.

The plan of the paper is as follows. In Section 2, we state a number of preliminary results. Then in Section 3 we prove Theorem 1. Finally, in Section 4 we prove Theorem 2.

2. Preliminary results

The following lemma is the local version of a well-known lemma of Pang and Zalcman [11, Lemma 2]. For a proof see [4, Lemma 2], also cf. [9, Lemma 2], [14, pp. 216–217], [7, pp. 299–300], [8, p. 4].

Lemma 1. *Let $\mathcal{F}$ be a family of functions meromorphic in a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$, such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at $z_0 \in D$, there exist, for each $0 \leq \alpha \leq k$,*

(a) *points $z_n \to z_0$,*
(b) *functions $f_n \in \mathcal{F}$, and*
(c) *positive numbers $\rho_n \to 0^+$*

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + f_n\zeta) \overset{\chi}{\Rightarrow} g(\zeta)$ on $\mathbf{C}$, where g is a nonconstant meromorphic function on $\mathbf{C}$, such that for every $\zeta \in \mathbf{C}$, $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$.

Lemma 2. [1, Lemma 5] *Let f be a nonconstant entire function of order ρ, $0 \leq \rho \leq 1$, all of whose zeros have multiplicity at least k, where $k \neq 2$ is a positive integer. And let $a \neq 0$ be a constant. If $\overline{E}_f(0) \subset \overline{E}_{f^{(k)}}(a) \subset \overline{E}_{f^{(k+1)}}(0)$, then*

$$f(z) = \frac{a(z-b)^k}{k!},$$

where b is a constant.

Lemma 3. [1, Lemma 6] *Let f be a nonconstant entire function of order ρ, $0 \leq \rho \leq 1$, all of whose zeros are multiple. Let $s \geq 4$ be an even integer and $a \neq 0$ be a constant. If $\overline{E}_f(0) \subset \overline{E}_{f''}(a) \subset \overline{E}_{f'''}(0) \cap \overline{E}_{f^{(s)}}(0)$, then*

$$f(z) = \frac{a(z-b)^2}{2},$$

where b is a constant.

Lemma 4. (see [2, pp. 118–119, 122–123]) *Let f be a meromorphic function on* $\mathbf{C}$. *If $f^{\#}$ is uniformly bounded on* $\mathbf{C}$, *then the order of f is at most 2. If f is an entire function, then the order of f is at most 1.*

The following lemma is a slight generalization of Theorem CFZ2 for sequences.

Lemma 5. (cf. [4, Lemma 5]) *Let $\{f_n\}$ be a sequence of functions holomorphic on a domain $D \subset \mathbf{C}$, all of whose zeros have multiplicity at least k, and let $\{h_n\}$ be a sequence of functions analytic on D such that $h_n(z) \Rightarrow h(z)$ on D, where $h(z) \neq 0$ for $z \in D$ and $k \neq 2$ be a positive integer. Suppose that, for each n, $f_n(z) = 0 \Longrightarrow f_n^{(k)}(z) = h_n(z)$ and $f_n^{(k)}(z) = h_n(z) \Longrightarrow f_n^{(k+1)}(z) = 0$. Then $\{f_n\}$ is normal on D.*

Proof. Suppose to the contrary that there exists $z_0 \in D$ such that $\{f_n\}$ is not normal in z_0. The convergence of $\{h_n\}$ to h implies that, in some neighborhood of z_0, we have $f_n(z) = 0 \Rightarrow |f_n^{(k)}(z)| \leq |h(z_0)| + 1$ (for large enough n). Thus we can apply Lemma 1 with $\alpha = k$ and A such that $kA + 1 > \max\Big\{|h(z_0)| + 1, \dfrac{|h(z_0)|}{(k-1)!}, \dfrac{k \cdot k!}{|h(z_0)|}\Big\} = \max\Big\{|h(z_0)| + 1, \dfrac{k \cdot k!}{|h(z_0)|}\Big\}$. So we can take an appropriate subsequence of $\{f_n\}$ (denoted also by $\{f_n\}$ after renumbering), together with points $z_n \to z_0$ and positive numbers $\rho_n \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k} \overset{\chi}{\Longrightarrow} g(\zeta) \quad \text{on } \mathbf{C},$$

where g is a nonconstant entire function and

$$g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1 = k(|h(z_0)| + 1) + 1.$$

We show that

$$\overline{E}_g(0) \subset \overline{E}_{g^{(k)}}(h(z_0)) \subset \overline{E}_{g^{(k+1)}}(0). \tag{10}$$

In fact, if there exists $\zeta_0 \in \mathbf{C}$, such that $g(\zeta_0) = 0$, then since $g(\zeta) \not\equiv 0$, there exist ζ_n, $\zeta_n \to \zeta_0$, such that if n is sufficiently large,

$$g_n(\zeta_n) = \frac{f_n(z_n + \rho_n \zeta_n)}{\rho_n^k} = 0.$$

Thus $f_n(z_n + \rho_n\zeta_n) = 0$, so that $f_n^{(k)}(z_n + \rho_n\zeta_n) = h_n(z_n + \rho_n\zeta_n)$, i.e., that $g_n^{(k)}(\zeta_n) = h_n(z_n+\rho_n\zeta_n)$. Since $g^{(k)}(\zeta_0) = \lim\limits_{n\to\infty} g_n^{(k)}(\zeta_n) = h(z_0)$, we have established that $\overline{E}_g(0) \subset \overline{E}_{g^{(k)}}(h(z_0))$.

Now, suppose there exists $\zeta_0 \in \mathbf{C}$, such that $g^{(k)}(\zeta_0) = h(z_0)$. If $g^{(k)}(\zeta) \equiv h(z_0)$, then $g^{(k+1)} \equiv 0$ and we are done. Thus we can assume that $g^{(k)}$ is not constant and since $f_n^{(k)}(z_n + \rho_n\zeta) - h_n(z_n + \rho_n\zeta) \Rightarrow g^{(k)}(\zeta) - h(z_0)$, we get by Hurwitz's Theorem that there exist ζ_n, $\zeta_n \to \zeta_0$, such that

$$f_n^{(k)}(z_n + \rho_n\zeta_n) - h_n(z_n + \rho_n\zeta_n) = g_n^{(k)}(\zeta_n) - h_n(z_n + \rho_n\zeta_n) = 0.$$

Thus we have $f_n^{(k+1)}(z_n + \rho_n\zeta_n) = 0$ and $g_n^{(k+1)}(\zeta_n) = 0$. Letting $n \to \infty$, we get that $g^{(k+1)}(\zeta_0) = 0$. This completes the proof of (10). Now, by Lemmas 4 and 2, we have

$g(\zeta) = \dfrac{h(z_0)(\zeta - \zeta_1)^k}{k!}$, where ζ_1 is a constant. Thus

$$g^\sharp(0) = \frac{|h(z_0)||\zeta_1|^{k-1}/(k-1)!}{1 + |h(z_0)|^2|\zeta_1|^{2k}/k!^2}.$$

Now, if $|\zeta_1| \leq 1$, then $g^\sharp(0) \leq \dfrac{|h(z_0)|}{(k-1)!} < kA + 1$, and if $|\zeta_1| > 1$, then $g^\sharp(0) \leq \dfrac{|h(z_0)||\zeta_1|^{k-1}/(k-1)!}{|h(z_0)|^2|\zeta_1|^{2k}/k!^2} \leq \dfrac{k \cdot k!}{|h(z_0)|} < kA+1$. In either case we get a contradiction. □

Similarly, we can get a slight generalization of Theorem CFZ3 for sequences.

Lemma 6. *Let $\{f_n\}$ be a sequence of functions holomorphic on a domain $D \subset \mathbf{C}$, all of whose zeros are multiple and $\{h_n\}$ be a sequence of functions analytic on D such that $h_n(z) \Rightarrow h(z)$ on D, where $h(z) \neq 0$ for $z \in D$, and $s \geq 2$ be an even integer. Suppose that, for each n, $f_n(z) = 0 \implies f_n''(z) = h_n(z)$ and $f_n''(z) = h_n(z) \implies f'''(z) = f_n^{(s)}(z) = 0$, then $\{f_n\}$ is normal on D.*

The proof is very similar to the proof of Lemma 5. We start to argue the same (with 2 instead of k), and then instead of proving (10) we prove that

$$\overline{E}_g(0) \subset \overline{E}_{g''}(h(z_0)) \subset \overline{E}_{g^{(3)}}(0) \cap \overline{E}_{g^{(s)}}(0).$$

The left inclusion is proved in the same manner. Concerning the right inclusion, we now deduce from $f_n''(z_n + \rho_n\zeta_n) - h_n(z_n + \rho_n\zeta_n) = 0$ that $f_n^{(3)}(z_n + \rho_n\zeta_n) = f_n^{(s)}(z_n + \rho_n\zeta_n) = 0$. Then, since $\rho_n f_n^{(3)}(z_n + \rho_n\zeta) \Rightarrow g^{(3)}(\zeta)$ in $\mathbf{C}$ and $\rho_n^{s-2} f_n^{(s)}(z_n + \rho_n\zeta) \Rightarrow g^{(s)}(\zeta)$ in $\mathbf{C}$, we conclude that $g^{(3)}(\zeta_0) = g^{(s)}(\zeta_0) = 0$. To get the final contradiction, we apply now Lemmas 4 and 3 instead of Lemmas 4 and 2.

The following result will play an essential role in treating transcendental functions which is used in the proofs of Theorems 2 and 3.

Theorem B. ([15], see also [2, p. 117]) *Let $f(z)$ be a function homomorphic in $\{z\colon R < |z| < \infty\}$, with essential singularity at $z = \infty$. Then $\overline{\lim\limits_{|z|\to\infty}} |z| f^{\#}(z) = +\infty$.*

For the proof of Theorem 2, we need also the following Lemma.

Lemma 7. *Let h be a holomorphic function on D, with a zero of order $\ell (\geq 1)$ at $z_0 \in D$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions with zeros of multiplicity at least k, such that $\{f_n\}$ and h satisfy conditions* (a) *and* (b) *of Theorem 2. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of nonzero numbers such that $\alpha_n \to 0$ as $n \to \infty$. Then $\{f_n(z_0 + \alpha_n\zeta)/\alpha_n^{k+\ell}\}_{n=1}^\infty$ is normal in $\mathbf{C}^*$.*

Proof. Without loss of generality, we may assume that $z_0 = 0$. In a neighborhood of the origin we have $h(z) = z^\ell b(z)$, where $b(z)$ is analytic, $b(0) \neq 0$. Define $r_n(\zeta) = \zeta^\ell b(\alpha_n\zeta)$. We will show that the assumptions of Lemma 5 hold in $\mathbf{C}^*$ for the sequences $\{G_n(\zeta)\}_{n=1}^\infty$, $G_n(\zeta) := f_n(\alpha_n\zeta)/\alpha_n^{k+\ell}$ and $\{r_n(\zeta)\}_{n=1}^\infty$. First, we have that $r_n(\zeta) \Rightarrow b(0)\zeta^\ell$ on $\mathbf{C}$ and $\zeta^\ell \neq 0$ in $\mathbf{C}^*$. Assume that $G_n(\zeta) = 0$. Then $f_n(\alpha_n\zeta) = 0$ and $f_n^{(k)}(\alpha_n\zeta) = (\alpha_n\zeta)^\ell b(\alpha_n\zeta)$, and we get that $G_n^{(k)}(\zeta) = r_n(\zeta)$. Suppose now that $G_n^{(k)}(\zeta) = r_n(\zeta)$. This means that $f_n^{(k)}(\alpha_n\zeta) = h(\alpha_n\zeta)$ and thus $f_n^{(k+1)}(\alpha_n\zeta) = 0$. We have $G_n^{(k+1)}(\zeta) = 0$, and thus the assumptions of Lemma 5 hold. Hence we deduce that $\{G_n(\zeta)\}$ is normal in $\mathbf{C}^*$, and the lemma is proved. □

The following lemma plays a similar role in the proof of Theorem 3 to the role of Lemma 7 in the proof of Theorem 2.

Lemma 8. *Let h be a holomorphic function on D, with a zero of order $\ell(\geq 1)$ at $z_0 \in D$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions whose zeros are multiple, such that $\{f_n\}$ and h satisfy conditions* (a) *and* (b) *of Theorem 3. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of nonzero numbers such that $\alpha_n \to 0$ as $n \to \infty$. Then $\{f_n(z_0+\alpha_n\zeta)/\alpha_n^{2+\ell}\}_{n=1}^{\infty}$ is normal in $\mathbf{C}^*$.*

The proof of this lemma is analogous to the proof of Lemma 7. Of course, we use Lemma 6 instead of Lemma 5.

3. Proof of Theorem 1

In this section, we do not use any of the preliminary results. The proof is elementary.

By Theorem CFZ1, $\mathcal{F}$ is normal at every point $z_0 \in D$ at which $h(z_0) \neq 0$ (so immediately we get that $\mathcal{F}$ is quasinormal). So let z_0 be a zero of h of order $\ell\,(\geq 1)$. Without loss of generality, we can assume that $z_0 = 0$, and then $h(z) = z^\ell b(z)$. Here b is an analytic function in $\Delta(0,\delta)$ and $b(z) \neq 0$ there. We assume that $0 < \delta < 1$, and by taking a subsequence and renumbering, we can assume that

$$f_n \Longrightarrow f \quad \text{in } \Delta'(0,\delta). \tag{11}$$

Now, if f is holomorphic in $\Delta'(0,\delta)$, we deduce by the maximum principle that $f_n \Rightarrow f$ on $\Delta(0,\delta)$, and we are done. So let us assume that $f_n \Rightarrow \infty$ in $\Delta'(0,\delta)$. Fix η, $0<\eta<\delta$. By the minimum principle (i.e., the maximum principle for $\{1/f_n\}$), there exists $N = N(\eta)$, such that for every $n \geq N$, f_n has $k_n(k_n \geq 1)$ simple zeros in $\overline{\Delta}(0,\eta) - \{0\}$, say $\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_{k_n}^{(n)}$ (otherwise we get that $f_n \Rightarrow \infty$ in $\Delta(0,\eta)$ and we are done). Since $f_n \Rightarrow \infty$ in $\Delta'(0,\delta)$, we get that

$$\max_{1\leq j\leq k_n} \{|\alpha_j^{(n)}|\} \to 0, \quad \text{as } n \to \infty. \tag{12}$$

We can write $f_n(z) = t_n(z)\prod_{i=1}^{k_n}\left(z-\alpha_i^{(n)}\right)$, where $t_n(z) \neq 0$ for $z \in \overline{\Delta}(0,\eta)$ and $n \geq N$. Since $\eta < 1$, we get by (12) that $\dfrac{t_n(z)}{b(z)} \Rightarrow \infty$ in $\overline{\Delta}(0,\eta)$. By condition (a) of Theorem 1, we have, for $n \geq N$, $f_n'(\alpha_j^{(n)}) = \alpha_j^{(n)\ell} b(\alpha_j^{(n)})$, $1 \leq j \leq k_n$. By calculation,

$$f_n'(z) = t_n'(z)\prod_{i=1}^{k_n}\left(z-\alpha_i^{(n)}\right) + t_n(z)\left[\prod_{i=1}^{k_n}\left(z-\alpha_i^{(n)}\right)\right]',$$

and so

$$t_n\left(\alpha_j^{(n)}\right)\left[\prod_{i=1}^{k_n}\left(z-\alpha_i^{(n)}\right)\right]'\Bigg|_{z=\alpha_j^{(n)}} = \alpha_j^{(n)\ell} b\left(\alpha_j^{(n)}\right). \tag{13}$$

Define, for $n \geq N$,

$$M_n(z) := \frac{t_n(z)}{b(z)}\left[\prod_{i=1}^{k_n}\left(z-\alpha_i^{(n)}\right)\right]' - z^\ell.$$

By (13) we get that $M_n\left(\alpha_j^{(n)}\right)=0$ for $1\le j\le k_n$, and so for $n\ge N$, M_n has at least k_n zeros in $\Delta'(0,\eta)$, including multiplicities. Here we use the fact that h has no common zero with any f_n. Since such a zero must be $z=0$ and would be a zero of order m (must be $m\ge 2$ by condition (a)) of f_n, and it would be a zero of order $m-1$ of M_n (if $\ell>m-1$) or even of order $\ell<m-1$ (if $\ell<m-1$), then we would not know that the number of zeros (including multiplicities) of M_n is at least k_n. This fact, under the assumption that there are no common zeros, will lead to the desired contradiction.

Claim 2. $\frac{t_n(z)}{b(z)}\left[\prod_{i=1}^{k_n}\left(z-\alpha_i^{(n)}\right)\right]'\Rightarrow\infty$ *in* $\Delta'(0,\eta)$.

Proof. We write

$$\frac{t_n(z)}{b(z)}\left[\prod_{i=1}^{k_n}\left(z-\alpha_i^{(n)}\right)\right]'=\sum_{j=1}^{k_n}\frac{t_n(z)}{b(z)}\prod_{i=1,i\ne j}^{k_n}\left(z-\alpha_i^{(n)}\right). \tag{14}$$

For any ε, $0<\varepsilon<\eta$, we have that

$$\frac{t_n(z)}{b(z)}\prod_{i=2}^{k_n}\left(z-\alpha_i^{(n)}\right)\Longrightarrow\infty\quad\text{in }\overline{R}_{\varepsilon,\eta}:=\{z\colon\varepsilon\le|z|\le\eta\}. \tag{15}$$

Indeed, $\frac{t_n(z)}{b(z)}\prod_{i=2}^{k_n}\left(z-\alpha_i^{(n)}\right)=\frac{f_n(z)}{b(z)\left(z-\alpha_1^{(n)}\right)}$, and since $\eta<1$ and by (11) and (12), this term tends uniformly to ∞ in $\overline{R}_{\varepsilon,\eta}$.

Now, for every j, $2\le j\le k_n$, we have that

$$\frac{\dfrac{t_n(z)}{b(z)}\prod_{i=2}^{k_n}\left(z-\alpha_i^{(n)}\right)}{\dfrac{t_n(z)}{b(z)}\prod_{i=1,i\ne j}^{k_n}\left(z-\alpha_i^{(n)}\right)}=\frac{z-\alpha_j^{(n)}}{z-\alpha_1^{(n)}},$$

and by (12) this term tends uniformly to 1 as $n\to\infty$. This means, that for every $1\le j\le k_n$ and $z\in\overline{R}_{\varepsilon,\eta}$, $\frac{t_n(z)}{b(z)}\prod_{i=1,i\ne j}^{k_n}\left(z-\alpha_i^{(n)}\right)$ lies in the same quarter plane, that is,

$$\begin{aligned}\Pi_{n,z}:=\Bigg\{z:\arg&\left[\frac{t_n(z)}{b(z)}\prod_{i=2}^{k_n}\left(z-\alpha_i^{(n)}\right)\right]-\frac{\pi}{4}\\&<\arg z<\arg\left[\frac{t_n(z)}{b(z)}\prod_{i=2}^{k_n}\left(z-\alpha_i^{(n)}\right)\right]+\frac{\pi}{4}\Bigg\},\end{aligned} \tag{16}$$

for large enough n.

Now, if a and b are two complex numbers in the same quarter plane, then $a+b$ also belongs to that quarter plane and $|a+b|\ge|a|,|b|$. We then conclude by (16) that for each $z \in \overline{R}_{\varepsilon,\eta}$, we have for large enough n,

$$\left| \frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} \left(z - \alpha_i^{(n)} \right) \right]' \right| \geq \left| \frac{t_n(z)}{b(z)} \prod_{i=2}^{k_n} \left(z - \alpha_i^{(n)} \right) \right|,$$

and by (15) and (14), Claim 2 is proved. □

Now, $\frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} \left(z - \alpha_i^{(n)} \right) \right]'$ has, for large enough n, exactly $k_n - 1$ zeros in $\Delta(0, \eta)$ (by Theorem Lu). Then for large enough n we have, for every z, $|z| = \eta$,

$$\left| M_n(z) - \frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} \left(z - \alpha_i^{(n)} \right) \right]' \right| = |z^\ell| < \left| \frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} \left(z - \alpha_i^{(n)} \right) \right]' \right|,$$

and by Rouche's Theorem, we get that M_n has $k_n - 1$ zeros in $\Delta(0, \eta)$, a contradiction. Theorem 1 is proved. □

4. Proof of Theorem 2

This proof is similar to the proof of Theorem 1 in [4]. By our Theorem 1, we need only to prove the case that $k \geq 3$. By Theorem CFZ2, $\mathcal{F}$ is normal at every point $z_0 \in D$ at which $h(z_0) \neq 0$ (so that $\mathcal{F}$ is quasinormal in D). Consider $z_0 \in D$ such that $h(z_0) = 0$. Without loss of generality, we can assume that $z_0 = 0$, and then $h(z) = z^\ell b(z)$, where $\ell\, (\geq 1)$ is an integer and $b(z) \neq 0$ is an analytic function in $\Delta(0, \delta)$. We take a subsequence $\{f_n\}_1^\infty \subset \mathcal{F}$, and we want to prove that $\{f_n\}$ is not normal at $z = 0$. Suppose by negation that $\{f_n\}$ is not normal at $z = 0$. Since $\{f_n\}$ is normal in $\Delta'(0, \delta)$, we can assume (after renumbering) that $f_n \Rightarrow F$ on $\Delta'(0, \delta)$. If $F(z) \not\equiv \infty$, then it is a holomorphic function. Hence, by the maximum principle, F extends to be analytic also at $z = 0$, and so $f_n \Rightarrow F$ on $\Delta(0, \delta)$, and we are done. Therefore, we assume that

$$f_n(z) \Longrightarrow \infty \quad \text{on } \Delta'(0, \delta). \tag{17}$$

Define $\mathcal{F}_1 = \left\{ F = \frac{f_n}{h} : n \in \mathbf{N} \right\}$. It is enough to prove that $\mathcal{F}_1$ is normal in $\Delta(0, \delta)$. Indeed, if (after renumbering) $\frac{f_n(z)}{h} \Rightarrow H(z)$ on $\Delta(0, \delta)$, then since $h \neq 0$ in $\Delta'(0, \delta)$, it follows from (17) that $H(z) \equiv \infty$ in $\Delta'(0, \delta)$, and thus $H(z) \equiv \infty$ also in $\Delta(0, \delta)$. In particular, $\frac{f_n}{h}(z) \neq 0$ on each compact subset of $\Delta(0, \delta)$ for large enough n. Since $h \neq 0$ on $\Delta'(0, \delta)$ and since $f_n(0) \neq 0$ for every $n \geq 1$ by assumptions of the theorem, we obtain $f_n(z) \neq 0$ on each compact subset of $\Delta(0, \delta)$ for large enough n. Then by the minimum principle, it follows from (17) that $f_n(z) \Rightarrow \infty$ on $\Delta(0, \delta)$, and this implies the normality of $\mathcal{F}$. So suppose to the contrary that $\mathcal{F}_1$ is not normal at $z = 0$. By Lemma 1 and the assumptions of Theorem 2, there exist (after renumbering) points $z_n \to 0$, $\rho_n \to 0^+$ and a nonconstant meromorphic function on $\mathbf{C}$, $g(\zeta)$ such that

$$g_n(\zeta) = \frac{F_n(z_n + \rho_n \zeta)}{\rho_n^k} = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k h(z_n + \rho_n \zeta)} \overset{\chi}{\Longrightarrow} g(\zeta) \quad \text{on } \mathbf{C}, \tag{18}$$

all of whose zeros have multiplicity at least k and

$$\text{for every } \zeta \in \mathbf{C}, \quad g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1, \tag{19}$$

where $A > 1$ is a constant. Here we have used Lemma 1 with $\alpha = k$. Observe that $g_n(z) = 0$ implies $g_n^{(k)}(\zeta) = 1$ and so A can be chosen to be any number such that $A \geq 1$. After renumbering we can assume that $\{z_n/\rho_n\}_{n=1}^{\infty}$ converges. We separate now into two cases.

Case (A).

$$\frac{z_n}{\rho_n} \to \infty. \tag{20}$$

Claim 3. (1) $g(\zeta) = 0 \Longrightarrow g^{(k)}(\zeta) = 1$; (2) $g^{(k)}(\zeta) = 1 \Longrightarrow g^{(k+1)}(\zeta) = 0$.

Proof. Observe that from (18) and the fact that $h(z) \neq 0$ in $\Delta'(0, \delta)$, it follows that g is an entire function. Suppose that $g(\zeta_0) = 0$. Since $g(\zeta) \not\equiv 0$, there exist $\zeta_n \to \zeta_0$, such that $g_n(\zeta_n) = 0$, and thus $f_n(z_n + \rho_n\zeta_n) = 0$. Since f_n and h has no common zeros, it follows by the assumption that ζ_n is a zero of multiplicity k of $g_n(\zeta)$. By Leibniz's rule, and condition (a) of Theorem 2, it follows that $g_n^{(k)}(\zeta_n) = 1$ and thus $g^{(k)}(\zeta_0) = 1$.

For the proof of the other part of Claim 3, observe first that by (20) we have

$$\frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k z_n^\ell} \Rightarrow g(\zeta) \quad \text{on } \mathbf{C},$$

and thus

$$\frac{f_n^{(k)}(z_n + \rho_n\zeta)}{z_n^\ell} \Rightarrow g^{(k)}(\zeta) \quad \text{on } \mathbf{C},$$

and then again by (19) we get that

$$\frac{f_n^{(k)}(z_n + \rho_n\zeta)}{h(z_n + \rho_n\zeta)} \Rightarrow g^{(k)}(\zeta) \quad \text{on } \mathbf{C}.$$

Thus, if there exists $\zeta_0 \in \mathbf{C}$, such that $g^{(k)}(\zeta_0) = 1$, there exists a sequence $\zeta_n \to \zeta_0$, such that $f_n^{(k)}(z_n + \rho_n\zeta_n) = h(z_n + \rho_n\zeta) \neq 0$. By assumption (b) of Theorem 2 we get that $f_n^{(k+1)}(z_n + \rho_n\zeta_n) = 0$, and letting n tend to ∞ we get that $g^{(k+1)}(\zeta_0) = 0$. Claim 3 is proved. □

We conclude by Lemmas 2 and 4 that $g(\zeta) = \dfrac{(\zeta - b)^k}{k!}$ for some $b \in \mathbf{C}$ (observe that g is holomorphic by (20)). By calculation we get that

$$g^\sharp(0) = \frac{|b|^{k-1}/(k-1)!}{1 + |b|^{2k}/k!^2}.$$

Then if $|b| \leq 1$, we get that $g^\sharp(0) \leq \dfrac{1}{(k-1)!}$, and if $|b| \geq 1$, then $g^\sharp(0) \leq \frac{k}{2}$. In either case, we get a contradiction to (19).

Case (B).

$$\frac{z_n}{\rho_n} \to \alpha \in \mathbf{C}. \tag{21}$$

As in Case (A), it follows that $g(\zeta_0) = 0 \Longrightarrow g^{(k)}(\zeta_0) = 1$. Now set

$$G_n(\zeta) = \frac{f_n(\rho_n\zeta)}{\rho_n^{k+\ell}}.$$

From (18) and (21) we have

$$G_n(\zeta) \Longrightarrow G(\zeta) = g(\zeta - \alpha)\zeta^\ell b(0) \quad \text{on } \mathbf{C}. \tag{22}$$

Indeed,

$$\frac{f_n(\rho_n\zeta)}{\rho_n^{k+\ell}} = \frac{f_n(\rho_n\zeta)}{\rho_n^k h(\rho_n\zeta)} \cdot \frac{h(\rho_n\zeta)}{\rho_n^\ell} = \frac{f_n\left(z_n + \rho_n\left(\zeta - \frac{z_n}{\rho_n}\right)\right)}{\rho_n^k h\left(z_n + \rho_n\left(\zeta - \frac{z_n}{\rho_n}\right)\right)} \frac{(\rho_n\zeta)^\ell b(\rho_n\zeta)}{\rho_n^\ell}$$

(cf. [12, p. 7]). Since g has a pole of order ℓ at $\zeta = -\alpha$ (here we use the fact that for every n, h has no common zeros with f_n) and since $\{G_n\}$ are analytic, we have

$$G(0) \neq 0, \infty. \tag{23}$$

We now consider several subcases, depending on the nature of G.

Case (BI). G is a polynomial. Since $\{f_n\}$ is not normal at $z = 0$, there exists (after renumbering) a sequence $z_n^* \to 0$ such that

$$f_n(z_n^*) = 0. \tag{24}$$

Otherwise, there is some δ', $0 < \delta' < \delta$ such that (before renumbering) $f_n(z) \neq 0$ in $\Delta(0, \delta')$, and since $f_n(z) \Rightarrow \infty$ on $\Delta'(0, \delta)$ we would have by the minimum principle that $f_n(z) \Rightarrow \infty$ on $\Delta(0, \delta)$, a contradiction to the non-normality of $\{f_n\}$ at $z = 0$. We have that all the zeros of g are of multiplicity exactly k. Then by (22) and (23), it follows that all the zeros of G are also of multiplicity exactly k. We consider now two possibilities.

Case (BI1). $\deg(G) = 0$. We can assume that z_n^* from (24) is the closest zero of f_n to the origin. Then we have

$$\frac{f_n(\rho_n\zeta)}{\rho_n^{k+\ell} b(\rho_n\zeta)} \Longrightarrow \frac{G(0)}{b(0)} \quad \text{on } \mathbf{C}. \tag{25}$$

By (25) we have

$$\frac{z_n^*}{\rho_n} \to \infty. \tag{26}$$

Define $t_n(\zeta) = f_n(z_n^*\zeta)/\left(z_n^{*k+\ell} b(z_n^*\zeta)\right)$. We want to show that $\{t_n(\zeta)\}$ is normal in $\mathbf{C}^*$. For this purpose set $\tilde{t}_n(\zeta) = f_n(z_n^*\zeta)/z_n^{*k+\ell}$. Since $b(0) \neq 0, \infty$ and $z_n^* \to 0$, the normality of $\{t_n\}$ is equivalent to the normality of $\{\tilde{t}_n\}$, and the latter follows by Lemma 7. Now, if $\{t_n\}$ is not normal at $\zeta = 0$, then we can write (after renumbering) $t_n(\zeta) \Rightarrow \infty$ on $\mathbf{C}^*$; but $t_n(1) = 0$, so this is not possible. Hence $\{t_n(\zeta)\}$ is normal at $\zeta = 0$. By (25) and (26), $t_n(0) \to 0$ as $n \to \infty$; and thus since $t_n(\zeta) \neq 0$ in $\Delta(0, 1/2)$, we get by Hurwitz's Theorem that $t_n(\zeta) \Rightarrow 0$ on $\mathbf{C}$. But $t_n(1) = 0$; so by assumption (b) of Theorem 2, we get that $t_n^{(k)}(1) = 1$, a contradiction.

Case (BI2). $G^{(k)} \equiv b(0)\zeta^\ell$. Then we have $G^{(k-1)}(\zeta) = \dfrac{b(0)\zeta^{\ell+1}}{\ell+1} + C$ and $G^{(k-2)}(\zeta) = \dfrac{b(0)\zeta^{\ell+2}}{(\ell+1)(\ell+2)} + C\zeta + D$, where C and D are two constants. Since all zeros of G have multiplicity exactly k, then for any zero $\widehat{\zeta}$ of G, we have $G^{(k-2)}(\widehat{\zeta}) =$

$G^{(k-1)}(\widehat{\zeta}) = 0$. So

$$\frac{\widehat{\zeta}^{\ell+1}}{\ell+1} + C = 0, \quad \text{and} \quad \frac{\widehat{\zeta}^{\ell+2}}{(\ell+1)(\ell+2)} + C\widehat{\zeta} + D = 0. \tag{27}$$

By calculation, we have $\frac{(\ell+1)C}{\ell+2}\widehat{\zeta} = -D$. If $CD = 0$, then by (27), $\widehat{\zeta} = 0$, a contradiction. If $CD \neq 0$, then $\widehat{\zeta} = -\frac{(\ell+2)D}{(\ell+1)C}$, which implies that G has only one zero ζ_0, and then

$$G(\zeta) = \frac{b(0)\zeta_0^\ell(\zeta-\zeta_0)^k}{k!}.$$

This contradicts $G^{(k)} \equiv b(0)\zeta^\ell$.

Case (BI3). G is a nonconstant polynomial and $G^{(k)} \not\equiv b(0)\zeta^\ell$. Since all zeros of G have multiplicity exactly k, we may assume that

$$G = A\prod_{j=1}^{t}(\zeta-\zeta_j)^k.$$

where $A \neq 0$ is a constant and $\zeta_j \neq 0$, $j = 1, 2, \cdots, t$.

Claim 4. $G(\zeta) = 0 \Longrightarrow G^{(k)}(\zeta) = b(0)\zeta^\ell \Longrightarrow G^{(k+1)}(\zeta) = 0$.

Proof. Suppose first that $G(\zeta_0) = 0$. Then there exists a sequence, $\zeta_n \to \zeta_0$, such that $f_n(\rho_n\zeta_n) = 0$, and thus $f_n^{(k)}(\rho_n\zeta_n) = (\rho_n\zeta_n)^\ell b(\rho_n\zeta_n)$, that is, $\frac{f_n^{(k)}(\rho_n\zeta_n)}{\rho_n^\ell} = \zeta_n^\ell b(\rho_n\zeta_n)$. In the last equation, the left hand side tends to $\zeta_0^\ell b(0)$ as $n \to \infty$. This proves the first part of Claim 4.

Suppose now that $G^{(k)}(\zeta_0) = b(0)\zeta_0^\ell$. Since $G^{(k)}(\zeta) \not\equiv b(0)\zeta^\ell$, there exists a sequence $\zeta_n \to \zeta_0$, such that $\frac{f_n^{(k)}(\rho_n\zeta_n)}{\rho_n^\ell} = \zeta_n^\ell b(\rho_n\zeta_n)$, that is, $f_n^{(k)}(\rho_n\zeta_n) = (\rho_n\zeta_n)^\ell b(\rho_n\zeta_n)$, and thus $f_n^{(k+1)}(\rho_n\zeta_n) = 0$. Since $\frac{f_n^{(k+1)}(\rho_n\zeta)}{\rho_n^{\ell-1}} \Rightarrow G^{(k+1)}(\zeta)$, we deduce that $G^{(k+1)}(\zeta_0) = 0$, and this completes the proof of the Claim 4. □

It follows from Claim 4 that $G^{(k+1)}(\zeta_j) = 0$, for $1 \leq j \leq t$.

If $t \geq 2$, we know that for every $1 \leq j \leq t$,

$$\begin{aligned} G^{(k+1)}(\zeta) &= A\left[\prod_{j=1}^{t}(\zeta-\zeta_j)^k\right]^{(k+1)} \\ &= A\left\{\sum_{\mu=0}^{k+1}\binom{k+1}{\mu}\left[(\zeta-\zeta_j)^k\right]^{(k+1-\mu)}\left[\prod_{i=1,i\neq j}^{t}(\zeta-\zeta_i)^k\right]^{(\mu)}\right\} \\ &= A\left\{(k+1)k!\left[\prod_{i=1,i\neq j}^{t}(\zeta-\zeta_i)^k\right]' + (\zeta-\zeta_j)P_j(\zeta)\right\}, \end{aligned}$$

where P_j is a polynomial. Thus, by Claim 4 we have

$$\left[\prod_{i=1, i\neq j}^{t} (\zeta-\zeta_i)^k\right]'\Bigg|_{\zeta_j} = 0, \quad 1 \le j \le t. \tag{28}$$

This means that for every $1 \le j \le t$,

$$\sum_{\substack{i=1\\ i\neq j}}^{t} (\zeta-\zeta_j)^{k-1} \prod_{\substack{\ell=1\\ \ell\neq i,j}}^{t} (\zeta-\zeta_\ell)^k \Big|_{\zeta_j} = 0.$$

Dividing in $\prod_{\ell\neq j}(\zeta_j-\zeta_\ell)^{k-1}$ gives

$$\sum_{\substack{i=1\\ i\neq j}}^{t} \prod_{\substack{\ell=1\\ \ell\neq i,j}}^{t} (\zeta_j-\zeta_\ell) = 0.$$

Thus $T''(\zeta_j) = 0$ for $1 \le j \le t$, where $T(\zeta) = \prod_{i=1}^{t}(\zeta-\zeta_i)$.

Now, if $t \ge 3$, then T'' is of degree $t-2$, and vanishes at t different points, a contradiction. If $t = 2$, we get from (28) that $\left[(\zeta-\zeta_2)^k\right]'\Big|_{\zeta_1} = 0$ and this is also a contradiction. So $t = 1$ and G has only one zero ζ_0 ($\zeta_0 \neq 0$), which means that $G(\zeta) = \dfrac{b(0)\zeta_0^\ell(\zeta-\zeta_0)^k}{k!}$.

By Hurwitz's Theorem, there exists a sequence $\zeta_{n,0} \to \zeta_0$, such that $G_n(\zeta_{n,0}) = 0$. If there exists δ', $0 < \delta' < \delta$, such that for every n (after renumbering), $f_n(z)$ has only one zero $z_{n,0} = \rho_n\zeta_{n,0}$ in $\Delta(0,\delta')$.

Set

$$H_n(z) = \frac{f_n(z)}{(z-z_{n,0})^k}.$$

Since $H_n(z)$ is a nonvanishing holomorphic function in $\Delta(0,\delta')$ and $H_n(z) \Rightarrow \infty$ on $\Delta'(0,\delta)$, we can deduce as before by the minimum principle that $H_n(z) \Rightarrow \infty$ on $\Delta(0,\delta')$. But

$$H_n(2z_{n,0}) = \frac{f_n(2z_{n,0})}{z_{n,0}^k} = \frac{\rho_n^\ell G_n(2\zeta_{n,0})}{\zeta_{n,0}^k} \to 0,$$

a contradiction. Thus, we can assume, after renumbering, that for every $\delta' > 0$, f_n has at least two zeros in $\Delta(0,\delta')$ for large enough n. Thus, there exists another sequence of points $z_{n,1} = \rho_n\zeta_{n,1}$, tending to zero, where $z_{n,1}$ is also a zero of $f_n(z)$ and $\zeta_{n,1} \to \infty$, as $n \to \infty$. We can also assume that $z_{n,1}$ is the closest zero to the origin of f_n, except $z_{n,0}$. Now set $c_n = z_{n,0}/z_{n,1}$ and define $K_n(\zeta) = f_n(z_{n,1}\zeta)/z_{n,1}^{k+\ell}$. By Lemma 7, $\{K_n(\zeta)\}$ is normal in $\mathbf{C}^*$. Now, if $\{K_n\}$ is normal at $\zeta = 0$, then after renumbering we can assume that

$$K_n(\zeta) \Longrightarrow K(\zeta) \quad \text{on } \mathbf{C}.$$

If $K(\zeta) \not\equiv$ const., then consider

$$L_n(\zeta) := \frac{K_n(\zeta)}{(\zeta-c_n)^k}.$$

Since $c_n \underset{n\to\infty}{\longrightarrow} 0$, then the sequence $\{L_n\}_1^\infty$ is normal in $\mathbf{C}^*$. It is also normal at $\zeta = 0$. Indeed, $K_n(c_n) = 0$ (a zero of order k) and so L_n is a nonvanishing holomorphic function in $\Delta(0,1)$. Thus (after renumbering)

$$L_n(\zeta) \Longrightarrow \frac{K(\zeta)}{\zeta^k} \quad \text{on } \mathbf{C}.$$

But

$$L_n(0) = \frac{K_n(0)}{(-c_n)^k} = \frac{G_n(0)}{\zeta_{n,1}^\ell(-\zeta_{n,0})^k} \underset{n\to\infty}{\longrightarrow} 0, \quad (\text{since } \zeta_{n,1} \underset{n\to\infty}{\longrightarrow} \infty),$$

and $L_n(\zeta) \neq 0$ in $\Delta(0,1/2)$; thus $K(\zeta)/\zeta^k \equiv 0$ in $\mathbf{C}$, a contradiction. If, on the other hand, $K(\zeta) \equiv$ const., then $K(\zeta) \equiv 0$ and $K^{(k)}(1) = 0$. But $K^{(k)}(1) = \lim_{n\to\infty} K_n^{(k)}(1) = \lim_{n\to\infty} \frac{f_n^{(k)}(z_{n,1})}{z_{n,1}^\ell} = \lim_{n\to\infty} \frac{h(z_{n,1})}{z_{n,1}^\ell} = \lim_{n\to\infty} b(z_{n,1}) = b(0)$, a contradiction. Hence we can deduce that $\{K_n\}$ is not normal at $\zeta = 0$, and since $K_n(\zeta)$ is holomorphic in Δ, then

$$K_n(\zeta) \Longrightarrow \infty \quad \text{on } \mathbf{C}^*.$$

But $K_n(1) = 0$, a contradiction.

Case (BII). $G(\zeta)$ is a transcendental entire function. Consider the family

$$\mathcal{F}(G) = \left\{ t_n(z) := \frac{G(2^n z)}{2^{n(k+\ell)}} : n \in \mathbf{N} \right\}.$$

By Claim 4, we deduce

(i) $t_n(z) = 0 \Longrightarrow t_n^{(k)}(z) = z^\ell$, and

(ii) $t_n^{(k)}(z) = z^\ell \Longrightarrow t_n^{(k+1)}(z) = 0$.

We then get by Theorem CFZ2 that $\mathcal{F}(G)$ is normal in $\mathbf{C}^*$. Thus there exists $M > 0$ such that for every $z \in R_{1,2} := \{z \colon 1 \le |z| \le 2\}$,

$$t_n^{\#}(z) = \frac{2^{n(k+\ell+1)}|G'(2^n z)|}{2^{2n(k+\ell)} + |G(2^n z)|^2} \le M.$$

Set $r(\zeta) := G(\zeta)/\zeta^{k+\ell}$. Then r is a transcendental meromorphic function, whose only pole is $\zeta = 0$. For every ζ, $|\zeta| \ge 2$ there exists $n \ge 1$ and $z \in R_{1,2}$, such that

$$\zeta = 2^n z. \tag{29}$$

Calculation gives

$$r^\sharp(\zeta) = \frac{|G'(\zeta)\zeta^{k+\ell} - (k+\ell)\zeta^{k+\ell-1}G(\zeta)|}{|\zeta|^{2(k+\ell)} + |G(\zeta)|^2}.$$

Thus, if $|\zeta| \ge 2$ satisfies (29), then

$$\begin{aligned} |\zeta r^\sharp(\zeta)| &= |2^n z| \frac{|G'(2^n z)(2^n z)^{k+\ell} - (k+\ell)(2^n z)^{k+\ell-1}G(2^n z)|}{|2^n z|^{2(k+\ell)} + |G(2^n z)|^2} \\ &\le \frac{2^{k+\ell+1}\cdot 2^{n(k+\ell+1)}|G'(2^n z)|}{2^{2n(k+\ell)} + |G(2^n z)|^2} + \frac{(k+\ell)2^{(n+1)(k+\ell)}|G(2^n z)|}{2^{2n(k+\ell)} + |G(2^n z)|^2}. \end{aligned} \tag{30}$$

By separating into two cases, depending on $|G(2^n z)| > 2^{(n+1)(k+\ell)}$ or $|G(2^n z)| \le 2^{(n+1)(k+\ell)}$, we see that the last expression in (30) is less or equal to

$$2^{k+\ell+1} t_n^\sharp(z) + (k+\ell)2^{2(k+\ell)}.$$

Thus, to every $|\zeta| \geq 2$,

$$|\zeta r^\sharp(\zeta)| \leq M \cdot 2^{k+\ell+1} + (k+\ell)2^{2(k+\ell)}.$$

But, according to Theorem B, $\overline{\lim}_{\zeta\to\infty} |\zeta| r^\sharp(\zeta) = \infty$, and we thus have a contradiction (cf. [3, pp. 19–21]). Theorem 2 is proved. □

5. Proof of Theorem 3

By Theorem CFZ3, $\mathcal{F}$ is normal at every point $z_0 \in D$ at which $h(z_0) \neq 0$ (so that $\mathcal{F}$ is quasinormal in D). Consider $z_0 \in D$ such that $h(z_0) = 0$. Without loss of generality, we can assume that $z_0 = 0$, and then $h(z) = z^\ell b(z)$, where $\ell(\geq 1)$ is an integer and $b(z) \neq 0$ is an analytic function in $\Delta(0,\delta)$. We take a subsequence $\{f_n\}_1^\infty \subset \mathcal{F}$, and we only need to prove that $\{f_n\}$ is not normal at $z = 0$.

Define $\mathcal{F}_2 = \left\{F = \frac{f_n}{h} : n \in \mathbf{N}\right\}$. It is enough to prove that $\mathcal{F}_2$ is normal in $\Delta(0,\delta)$. Suppose to the contrary that $\mathcal{F}_2$ is not normal at $z = 0$. By Lemma 1 and the assumptions of Theorem 3, there exist (after renumbering) points $z_n \to 0$, $\rho_n \to 0^+$ and a nonconstant meromorphic function on $\mathbf{C}$, $g(\zeta)$ such that

$$g_n(\zeta) = \frac{F_n(z_n + \rho_n\zeta)}{\rho_n^2} = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^2 h(z_n + \rho_n\zeta)} \overset{\chi}{\Longrightarrow} g(\zeta) \quad \text{on } \mathbf{C}, \tag{31}$$

all of whose zeros are multiple and

$$\text{for every } \zeta \in \mathbf{C}, \quad g^\sharp(\zeta) \leq g^\sharp(0) = 2A + 1, \tag{32}$$

where $A > 1$ is a constant. After renumbering we can assume that $\{z_n/\rho_n\}_{n=1}^\infty$ converges. We separate now into two cases.

Case (A). $\frac{z_n}{\rho_n} \to \infty$. Similar to the proof of Theorem 2, we can prove that $g(\zeta) = 0 \Longrightarrow g''(\zeta) = 1$ and that $g''(\zeta) = 1 \Longrightarrow g'''(\zeta) = g^{(s)}(\zeta) = 0$. Then by Lemmas 4 and 3, we have

$$g(\zeta) = \frac{(\zeta - b)^2}{2},$$

for some $b \in \mathbf{C}$. Thus $g^\sharp(0) = \dfrac{|b|}{1 + |b|^4/4}$ and then $g^\sharp(0) \leq 1$, which contradicts (32).

Case (B).

$$\frac{z_n}{\rho_n} \to \alpha \in \mathbf{C}. \tag{33}$$

As in the proof of Theorem 2, we have $g(\zeta_0) = 0 \Longrightarrow g''(\zeta_0) = 1$. Now set $G_n(\zeta) = \dfrac{f_n(\rho_n\zeta)}{\rho_n^{2+\ell}}$. From (31) and (33) we have

$$G_n(\zeta) \Longrightarrow G(\zeta) = b(0)g(\zeta - \alpha)\zeta^\ell \quad \text{on } \mathbf{C}.$$

Since g has a pole of order ℓ at $\zeta = -\alpha$, $G(0) \neq 0, \infty$.

We now consider several subcases, depending on the nature of G.

Case (BI). G is a polynomial. By a similar method of proof used in the proof of Theorem 2 (and using Lemma 8 instead of Lemma 7 in the appropriate places), we can get

$$G(\zeta) = \frac{b(0)\zeta_0^\ell(\zeta - \zeta_0)^2}{2},$$

and also we can arrive at a contradiction.

Case (BII). $G(\zeta)$ is a transcendental entire function. Consider the family

$$\mathcal{F}(G) = \left\{ t_n(z) := \frac{G(2^n z)}{2^{n(2+\ell)}} : n \in \mathbf{N} \right\}.$$

We have

(i) $t_n(z) = 0 \Longrightarrow t_n''(z) = z^\ell$, and

(ii) $t_n''(z) = z^\ell \Longrightarrow t_n'''(z) = t_n^{(s)}(z) = 0$.

We then get by Theorem CFZ3 that $\mathcal{F}(G)$ is normal in $\mathbf{C}^*$. Set $r(\zeta) := G(\zeta)/\zeta^{2+\ell}$, and we have that, for every ζ, $|\zeta| \geq 2$, there exists $n \geq 1$ and $z \in R_{1,2}$, such that

$$|\zeta r^\sharp(\zeta)| \leq M \cdot 2^{2+\ell+1} + (2+\ell)2^{2(2+\ell)}.$$

But, according to Theorem B, $\overline{\lim_{\zeta\to\infty}} |\zeta| r^\sharp(\zeta) = \infty$, and we thus have a contradiction (cf. [3, pp. 19–21]). Theorem 3 is proved. □

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