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UNIVERSAL TEICHMÜLLER SPACES AND $F(p,q,s)$ SPACE

Xiaogao Feng, Shengjin Huo and Shuan Tang*

China West Normal University, College of Mathematics and Information
Nanchong, 637002, P. R. China; and
Soochow University, Department of Mathematics
Suzhou, 215006, P. R. China; fengxiaogao603@163.com

Tianjin Polytechnic University, Department of Mathematics
Tianjin, 300387, P. R. China; sjhuo@pku.edu.cn

Guizhou Normal University, School of Mathematics Sciences
Guiyang 550001, P.R. China; tsaflyhigher@163.com

Abstract. In this paper, we introduce the $F(p,s)$-Teichmüller space and investigate its Schwarzian derivative model and pre-logarithmic derivative model. In particular, we prove that the pre-logarithmic derivative model is a disconnected subset of Besov type space $F(p,s)$ and the Bers projection is holomorphic.

1. Introduction

Let $\Delta = \{z\colon |z| < 1\}$ be the unit disk in the complex plane $\mathbf{C}$, $\Delta^* = \mathbf{C}\backslash\overline{\Delta}$ be the outside of the unit disk and $S^1 = \{z \in \mathbf{C}\colon |z| = 1\}$ be the unit circle. Let $\alpha > 0$, the Bloch-type space B^α consists of all holomorphic functions f on Δ such that

$$\|f\|_B = \sup_{z\in\Delta}(1-|z|^2)^\alpha |f'(z)| < \infty,$$

and the subspace B_0^α consists of all functions $f \in B^\alpha$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

We denote by $\mathrm{BMO}(S^1)$ the space of all integrable functions on S^1 such that

$$\|u\|_{BMO} = \sup_I \frac{1}{|I|}\int_I |u-u_I|\,d\theta < \infty, \tag{1}$$

where I is any arc on S^1, $|I|$ denotes the Lebesgue measure of I, and

$$u_I = \frac{1}{|I|}\int_I u\,d\theta \tag{2}$$

is the average of u over I. A holomorphic function f on Δ belongs to $\mathrm{BMOA}(\Delta)$ if and only if it is a Poisson integral of some function which belongs to $\mathrm{BMO}(S^1)$.

For any $a \in \Delta$, set $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$, $z \in \Delta$. For $p > 1$, $q > -2$ and $s \geq 0$, the space $F(p,q,s)$ consists of all holomorphic functions f on the unit disk Δ with the

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*The first and the corresponding author.

following finite norm

$$\|f\|_{F_{p,q,s}}^p = \sup_{a\in\Delta} \iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s\, dx\, dy < \infty, \tag{3}$$

and $F_0(p,q,s)$ consists of all functions $f \in F(p,q,s)$ with

$$\lim_{|a|\to 1} \iint_\Delta |f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s\, dx\, dy = 0. \tag{4}$$

The space $F(p,q,s)$ was introduced by Zhao [23]. It is well known that $F(p,q,s)$ is trivial if $q+s \le -1$. For $p \ge 1$, $F(p,q,s)$ is a Banach space contained in the Bloch-type space B^α and $F_0(p,q,s) \subset B_0^\alpha$ with $\alpha = \frac{q+2}{p}$. It is also known that $F(2,0,1)$ is the BMOA space (see [8]) and $F(2,0,s)$ is the Q_s space (see [19, 20]). In this paper, we shall mainly concentrate on the Besov type space $F(p,s) = F(p,p-2,s)$ and $F_0(p,s) = F_0(p,p-2,s)$.

Let $\Omega = \Delta$ or $\Omega = \Delta^*$. Given an arc I of the unit circle S^1, the Carleson box is defined by

$$S_\Omega(I) = \begin{cases} \{z \in \Delta\colon 1-|I| \le |z| < 1, z/|z| \in I\}, & \Omega = \Delta, \\ \{z \in \Delta^*\colon 1 \le |z| < 1+|I|, z/|z| \in I\}, & \Omega = \Delta^*. \end{cases}$$

Let $s > 0$. A positive measure λ on Ω is called an s-Carleson measure if

$$\|\lambda\|_{C,s} = \sup_{I\subset S^1} \frac{\lambda(S_\Omega(I))}{|I|^s} < \infty,$$

and a compact s-Carleson measure if

$$\lim_{|I|\to 0} \frac{\lambda(S_\Omega(I))}{|I|^s} = 0.$$

We denote by $CM_s(\Omega)$ the set of all s-Carleson measures on Ω and $CM_{s,0}(\Omega)$ the set of all compact s-Carleson measures on Ω. 1-Carleson measure is the classical Carleson measure. By [19], we know that a positive measure λ on Δ is an s-Carleson measure if and only if

$$\sup_{a\in\Delta} \iint_\Delta \left(\frac{1-|a|^2}{|\,1-\overline{a}z\,|^2}\right)^s d\lambda(z) < \infty, \tag{5}$$

and is a compact s-Carleson measure if and only if

$$\lim_{|a|\to 1} \iint_\Delta \left(\frac{1-|a|^2}{|\,1-\overline{a}z\,|^2}\right)^s d\lambda(z) = 0. \tag{6}$$

Let f be a quasiconformal mapping of the complex plane $\mathbf{C}$ onto itself. Then f is a homeomorphism with locally integral distributional derivatives, and satisfies the Beltrami equation $f_{\overline{z}} = \mu f_z$ with $\|\mu\|_\infty = \sup_{z\in\mathbf{C}} |\mu(z)| < 1$. Here we use the notations

$$f_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f, \quad f_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f.$$

This function μ is called the complex dilatation of f. The measurable Riemann mapping theorem (see [1]) says that for each measurable function μ on the complex plane $\mathbf{C}$ with $\|\mu\|_\infty < 1$, there is a quasiconformal mapping f on $\mathbf{C}$ with complex dilatation μ and f is unique up to a Möbius transformation of $\mathbf{C}$ onto itself.

A homeomorphism h is said to be quasisymmetric if there is some $M(h) > 0$ such that $|h(I^*)| \le M(h)|h(I)|$ for any interval $I \subset S^1$ with $|I| \le \pi$, where I^* is the interval with the same center as I but with double length. Denote by

$QS(S^1)$ the group of quasisymmetric homeomorphisms of the unit circle S^1. It is well known that a sense preserving self-homeomorphism h is quasisymmetric if and only if it can be extended to a quasiconformal self-homeomorphism of the unit disk Δ (see [3]). Douady and Earle [7] also gave a quasiconformal extension of h to the unit disk which is conformally invariant. Let $\text{Möb}(S^1)$ be the group of all Möbius transformations of Δ onto itself. The universal Teichmüller space is the right coset space $T = QS(S^1)/\text{Möb}(S^1)$.

Let $M(\Delta^*)$ denote the unit ball of the Banach space $L^\infty(\Delta^*)$ of all bounded measurable functions on Δ^*. For any $\mu \in M(\Delta^*)$, there exists a unique quasiconformal mapping f_μ of $\mathbf{C}$ whose complex dilatation is equal to μ in Δ^* and is zero in Δ. We normalize f_μ by

$$f_\mu(0) = f'_\mu(0) - 1 = f''_\mu(0) = 0.$$

We say that two Beltrami coefficients μ_1 and μ_2 in $M(\Delta^*)$ are Teichmüller equivalent, which is denoted by $\mu_1 \sim \mu_2$, if $f_{\mu_1}(\Delta) = f_{\mu_2}(\Delta)$. The universal Teichmüller space T can be described as $T = M(\Delta^*)/\sim$. We denote by $[\mu]$ the equivalent class containing $\mu \in M(\Delta^*)$.

Let S_Q be the class of all univalent analytic functions f in the unit disk Δ with the normalized condition $f(0) = f'(0) - 1 = 0$ that can be extended to a quasiconformal mapping in the whole plane. Set $T(1) = \{\log f' \colon f \text{ belongs to } S_Q\}$. It is known that $T(1)$ is an alternative model called pre-logarithmic derivative model of the universal Teichmüller space. $T(1)$ is a disconnected subset of the Bloch space B^1. Furthermore, $T_b = \{\log f' \in T(1) \colon f(\Delta) \text{ is bounded}\}$ and $T_\theta = \{\log f' \in T(1) \colon f(e^{i\theta}) = \infty\}$, $\theta \in [0, 2\pi)$, are the connected components of $T(1)$ (see [24]).

A quasisymmetric homeomorphism h is called strongly quasisymmetric homeomorphism if for each $\varepsilon > 0$, there is a constant $\delta > 0$ such that $|E| \le \delta |I|$ implies that $|h(E)| \le \varepsilon |h(I)|$, where $I \subset S^1$ is an interval and $E \subset I$ is a measurable subset. In other words, h is absolutely continuous and $\log h' \in BMO(S^1)$. It is equivalent to say that there exists a quasiconformal extension of h to Δ such that its complex dilatation $\mu(z)$ satisfies

$$\frac{|\mu(z)|^2}{1 - |z|^2}\, dx\, dy \in CM_1(\Delta)$$

(see [2]). The Teichmüller space called BMO-Teichmüller space with respect to the strongly quasisymmetric homeomorphisms has been much studied in recently years (see [6], [16]). In particular, Astala and Zinsmeister [2] proved that the pre-logarithmic derivative model $BMOA \cap T(1)$ of the BMO-Teichmüller space is disconnected open subset of BMOA.

Recently, Wulan and Ye [18] introduced the Q_K-Teichmüller space and showed that its pre-logarithmic derivative model is also disconnected subset of the Q_K space.

We denote by $N(p, s)$ the space of all holomorphic functions f on Δ with the following finite norm

$$\|f\|^p_{N_{p,s}} = \sup_{a \in \Delta} \iint_\Delta |f(z)|^p (1 - |z|^2)^{s+2p-2} \frac{(1 - |a|^2)^s}{|1 - \overline{a}z|^{2s}}\, dx\, dy. \tag{7}$$

We say an analytic function f belongs to $N_0(p, s)$ if $f \in N(p, s)$ and

$$\lim_{|a| \to 1} \iint_\Delta |f(z)|^p (1 - |z|^2)^{s+2p-2} \frac{(1 - |a|^2)^s}{|1 - \overline{a}z|^{2s}}\, dx\, dy = 0. \tag{8}$$

Zorboska [25] obtained a characterization of the relationship between the pre-logarithmic derivative $\log f'$ in space $F(p,s)$ and the Schwarzian derivative S_f in space $N(p,s)$.

Theorem A. [25] *Let f be conformal on Δ, $0 \le s < \infty$ and $1 \le p < \infty$. Then $\log f' \in F(p,s)$ if and only if $S_f \in N(p,s)$, while $\log f' \in F_0(p,s)$ if and only if $S_f \in N_0(p,s)$.*

It should be pointed out that the $F(2,1)$ case was proved by Astala and Zinsmeister in [2] and the $F(2,s)$ case was proved by Pau and Peláez in [13].

In this paper, we introduce the $F(p,s)$-Teichmüller space and investigate its Schwarzian derivative model and pre-logarithmic derivative model. In what follows, we always assume that $p \ge 2$ and $s > 0$. Denote by $M_{p,s}(\Omega)$ the Banach space of all essentially bounded measurable functions μ each of which induces an s-Carleson measure $\lambda_\mu(z) := \frac{|\mu(z)|^p}{|1-|z|^2|^{2-s}}\, dx\, dy \in CM_s(\Omega)$. The norm of $\mu \in M_{p,s}(\Omega)$ is defined as

$$\| \mu \|_s = \| \mu \|_\infty + \|\lambda_\mu\|_{C,s}^{1/p}, \tag{9}$$

where $\|\lambda_\mu\|_{C,s}$ is the s-Carleson norm of λ_μ on Ω. $M_{p,s,0}(\Omega)$ is the subspace of $M_{p,s}(\Omega)$ which consists of all elements μ such that $\lambda_\mu(z) \in CM_{s,0}(\Omega)$. Set $M^1_{p,s}(\Omega) = M_{p,s}(\Omega) \cap M(\Omega)$ and $M^1_{p,s,0}(\Omega) = M_{p,s,0}(\Omega) \cap M(\Omega)$, where $M(\Omega)$ denotes the unit ball of the Banach space $L^\infty(\Omega)$ of all bounded measurable functions on Ω. We define the $F(p,s)$-Teichmüller space $T_{F(p,s)}$ as $T_{F(p,s)} = M^1_{p,s}(\Delta^*)/\sim$ and the $F_0(p,s)$-Teichmüller space $T_{F_0(p,s)}$ as $T_{F_0(p,s)} = M^1_{p,s,0}(\Delta^*)/\sim$.

It is noted that $F(2,1)$-Teichmüller space is the BMO-Teichmüller space and the limit case $F(2,0)$-Teichmüller space is the Weil–Petersson Teichmüller space (see [5]) which has been much investigated in recently years and has wide applications to various areas such as mathematical physics, differential equation and computer vision. The limit case $F(p,0)$-Teichmüller space is the p-integrable Teichmüller space (see [9, 17, 21]).

The pre-logarithmic derivative model $\widetilde{T}_{F(p,s)}$ of $F(p,s)$-Teichmüller space is the set of functions $\log f'$, where f is conformal on Δ and admits a quasiconformal extension to the whole plane $\mathbf{C}$ such that its complex dilatation μ satisfies

$$\frac{| \mu(z) |^p}{(| z |^2 - 1)^{2-s}}\, dx\, dy \in CM_s(\Delta^*). \tag{10}$$

In this paper, we shall prove

Theorem 1.1. *Let $p \ge 2$ and $0 < s < 2$. $\widetilde{T}_{F(p,s)}$ is a disconnected subset of the space $F(p,s)$. Furthermore, $\widetilde{T}_b = \{\log f' \in \widetilde{T}_{F(p,s)} \colon f(\Delta)$ is bounded $\}$ and $\widetilde{T}_\theta = \{\log f' \in \widetilde{T}_{F(p,s)} \colon f(e^{i\theta}) = \infty\}$, $\theta \in [0, 2\pi)$, are the connected components of $\widetilde{T}_{F(p,s)}$.*

Let $B_\infty(\Delta)$ denote the Banach space of all holomorphic functions on Δ with norm

$$\|\varphi\|_{B_\infty} = \sup_{z \in \Delta} |\varphi(z)|(1 - |z|^2)^2 < \infty.$$

The Schwarzian derivative S_f of a conformal mapping f on Δ is defined as

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

The Bers projection $\Phi\colon M(\Delta^*) \to B_\infty(\Delta)$ is defined by $\mu \mapsto S_{f_\mu}$. The holomorphy of the Bers projection is important in the theory of Teichmüller space. It is known that $\Phi\colon M(\Delta^*) \to B_\infty(\Delta)$ is holomorphic and descends down to a mapping $B\colon T \to B_\infty(\Delta)$ known as the Bers embedding. Via the Bers embedding, T carries a natural complex Banach manifold structure so that B is a holomorphic split submersion. For the Bers projection Φ on $M^1_{p,s}(\Delta^*)$, we also obtain the following result.

Theorem 1.2. *Let $p \geq 2$ and $0 < s < 2$. The Bers projection $\Phi\colon M^1_{p,s}(\Delta^*) \to N(p,s)$ is holomorphic.*

Fix $z_0 \in \Delta^*$. For $\mu \in M^1_{p,s}(\Delta^*)$, let $g^{z_0}_\mu$ (abbreviated to be g_μ) be the quasiconformal mapping on the complex plane $\mathbf{C}$ whose complex dilatation equals to μ in Δ^* and zero in Δ, normalized by $g_\mu(0) = g'_\mu(0) - 1 = 0$, $g_\mu(z_0) = \infty$. Consider the pre-Bers projection mapping L_{z_0} on $M^1_{p,s}(\Delta^*)$ by setting $L_{z_0}(\mu) = \log g'_\mu$. Then $\bigcup_{z_0\in\Delta^*} L_{z_0}(M^1_{p,s}(\Delta^*)) = \widetilde{T}_{F(p,s)} \cap F(p,s)^0$, where $F(p,s)^0$ consists of all functions $\varphi \in F(p,s)$ with $\varphi(0) = 0$.

Theorem 1.3. *Let $p \geq 2$ and $0 < s < 2$. For $z_0 \in \Delta^*$, the pre-Bers projection mapping $L_{z_0}\colon M^1_{p,s}(\Delta^*) \to F(p,s)^0$ is holomorphic.*

Let f be a conformal mapping on Δ. The Grunsky kernel function is defined as

$$U(f,\zeta,z) = \frac{f'(\zeta)f'(z)}{(f(\zeta)-f(z))^2} - \frac{1}{(\zeta-z)^2}, \quad (\zeta,z) \in \Delta\times\Delta. \tag{11}$$

Let h be a quasisymmetric homeomorphism on the unit circle S^1. Then a kernel function induced by h is defined as

$$\phi_h(\zeta,z) = \frac{1}{2\pi i}\int_{S^1} \frac{h(w)}{(1-\zeta w)^2(1-zh(w))}\,dw, \quad (\zeta,z) \in \Delta\times\Delta. \tag{12}$$

These two kernel functions induce two functions,

$$U(f,z) = \left(\frac{1}{\pi}\iint_\Delta |U(f,\zeta,z)|^2\,d\xi\,d\eta\right)^{\frac12}, \quad z \in \Delta \tag{13}$$

and

$$\phi_h(z) = \left(\frac{1}{\pi}\iint_\Delta |\phi_h(\zeta,z)|^2\,d\xi\,d\eta\right)^{\frac12}, \quad z \in \Delta. \tag{14}$$

The functions $U(f,z)$ and $\phi_h(z)$ are important in Teichmüller theory (see [10], [15], [16]). They were used to characterize when a quasisymmetric homeomorphism is symmetric in [10] or belongs to the Weil–Petersson class in [15]. They were also used to study the BMO-Teichmüller theory in [16].

For any quasisymmetric homeomorphism h, there exists a unique pair of conformal mappings $f \in S_Q$ on Δ and g on Δ^*, such that $f(0) = f'(0) - 1 = 0$, $g(\infty) = \infty$ and $h = f^{-1}\circ g$ on S^1. We call this a normalized decomposition of h. Conversely, for each $f \in S_Q$, there exists a quasisymmetric homeomorphism h with normalized decomposition $h = f^{-1}\circ g$ on S^1 (see [11]). We have the following result.

Theorem 1.4. *Let $p \geq 2$, $0 < s < 2$ and h be a sense-preserving quasisymmetric homeomorphism with normalized decomposition $h = f^{-1}\circ g$. Then the following statements are equivalent:*

(1) $\log f' \in F_0(p,s)$;
(2) $|S_f(z)|^p(1-|z|^2)^{2p+s-2}\,dx\,dy \in CM_{s,0}(\Delta)$;

(3) *f can be extended to a quasiconformal mapping to the whole plane such that its complex dilatation μ satisfies $\frac{|\mu(z)|^p}{(|z|^2-1)^{2-s}}\,dx\,dy \in CM_{s,0}(\Delta^*)$;*
(4) $|U(f,z)|^p(1-|z|^2)^{p+s-2}\,dx\,dy \in CM_{s,0}(\Delta)$;
(5) $|\phi_h(\overline{z})|^2(1-|z|^2)^{p+s-2}\,dx\,dy \in CM_{s,0}(\Delta)$.

In what follows, $C(\cdot)$ will denote constant that depends only on the elements put in the bracket.

2. Bers projection and pre-Bers projection

In order to prove Theorem 1.2 and Theorem 1.3, we need some lemmas as follows.

Lemma 2.1. [22] *Suppose that $k > -1$, $r, t > 0$, and $r+t-k > 2$. If $t < k+2 < r$, then there exists a universal constant $C > 0$ such that for all $z, \zeta \in \Delta$,*

$$\iint_\Delta \frac{(1-\mid w \mid^2)^k}{\mid 1-\overline{w}z \mid^r \mid 1-\overline{w}\zeta \mid^t}\,du\,dv \le C\frac{(1-\mid z \mid^2)^{2+k-r}}{\mid 1-\overline{\zeta}z \mid^t},$$

where $w = u+iv$.

Lemma 2.1. *Let $\alpha > 0$, $\beta > 0$ and $s < 1+\alpha/2$. For a positive measure λ on Δ, set*

$$\widetilde{\lambda}(z) = \iint_\Delta \frac{(1-|z|^2)^\alpha(1-|w|^2)^\beta}{|1-\overline{z}w|^{\alpha+\beta+2}}\lambda(w)\,du\,dv.$$

If $\lambda \in CM_s(\Delta)$, then $\widetilde{\lambda} \in CM_s(\Delta)$ and there exists a constant $C' > 0$ such that

$$\|\widetilde{\lambda}\|_{C,s} \le C'\|\lambda\|_{C,s},$$

while $\widetilde{\lambda} \in CM_{s,0}(\Delta)$ if $\lambda \in CM_{s,0}(\Delta)$.

Proof. Set $k = \alpha$, $r = \alpha+\beta+2$, $t = 2s$ and note that $s < 1+\alpha/2$, it follows from Lemma 2.1 that there exists a universal constant $C > 0$ such that for any $a, w \in \Delta$,

$$\iint_\Delta \frac{(1-|z|^2)^\alpha}{|1-\overline{z}w|^{\alpha+\beta+2}|1-\overline{a}z|^{2s}}\,dx\,dy \le C\frac{(1-|w|^2)^{-\beta}}{|1-\overline{a}w|^{2s}}. \tag{15}$$

By Lemma 4.1.1 in Xiao [19], there exist some constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\|\lambda\|_{C,s} \le \sup_{a\in\Delta}\iint_\Delta \lambda(z)\frac{(1-\mid a \mid^2)^s}{\mid 1-\overline{a}z \mid^{2s}}\,dx\,dy \le C_2\|\lambda\|_{C,s}. \tag{16}$$

Consequently, we conclude from (15) that

$$\begin{aligned}
&\iint_\Delta \widetilde{\lambda}(z)\frac{(1-\mid a \mid^2)^s}{\mid 1-\overline{a}z \mid^{2s}}\,dx\,dy \\
&= \iint_\Delta \Big(\iint_\Delta \frac{(1-|z|^2)^\alpha(1-|w|^2)^\beta}{|1-\overline{z}w|^{\alpha+\beta+2}}\lambda(w)\,du\,dv\Big)\frac{(1-\mid a \mid^2)^s}{\mid 1-\overline{a}z \mid^{2s}}\,dx\,dy \\
&= \iint_\Delta \frac{(1-\mid a \mid^2)^s}{(1-|w|^2)^{-\beta}}\lambda(w)\,du\,dv \iint_\Delta \frac{(1-|z|^2)^\alpha}{|1-\overline{z}w|^{\alpha+\beta+2}|1-\overline{a}z|^{2s}}\,dx\,dy \\
&\le C\iint_\Delta \lambda(w)\frac{(1-\mid a \mid^2)^s}{\mid 1-\overline{a}w \mid^{2s}}\,du\,dv.
\end{aligned} \tag{17}$$

If $\lambda \in CM_s(\Delta)$, then from (16) and (17), we deduce that $\widetilde{\lambda} \in CM_s(\Delta)$ and there is a constant $C' = \frac{CC_2}{C_1}$ such that $\|\widetilde{\lambda}\|_{C,s} \le C'\|\lambda\|_{C,s}$. If $\lambda \in CM_{s,0}(\Delta)$, then

$$\lim_{|a|\to 1} \iint_\Delta \lambda(w) \left(\frac{1-|a|^2}{|\,1-\overline{a}w\,|^2}\right)^s du\,dv = 0. \tag{18}$$

We deduce from (17) that

$$\lim_{|a|\to 1} \iint_\Delta \widetilde{\lambda}(z) \left(\frac{1-|a|^2}{|\,1-\overline{a}z\,|^2}\right)^s dx\,dy = 0. \tag{19}$$

Therefore $\widetilde{\lambda} \in CM_{s,0}(\Delta)$. □

Let f be a conformal mapping on Δ^*. For any $z \in \Delta^*$, set

$$\beta_z(w) = \frac{1+wz}{w+\overline{z}} \quad \text{and} \quad \gamma_f(w) = \frac{(|z|^2-1)}{w-f(z)} f'(z). \tag{20}$$

Then β_z is an automorphism of Δ^* sending ∞ to z. We need a representation theorem of the Schwarzian derivative, which is proved by Astala and Zinsmeister [2].

Lemma 2.3. [2] *Let f be a conformal mapping on Δ^* and admits a quasiconformal extension to the whole plane, then for any $z \in \Delta^*$ and $w = u + iv$,*

$$S_f(z) = -\frac{3(|z|^2-1)^{-2}}{2\pi} \iint_\Delta \overline{\partial} g(w)\,du\,dv, \tag{21}$$

where $g = \gamma_f \circ f \circ \beta_z$.

We first show that the Bers projection is well defined.

Proposition 2.4. *Let $p \ge 2$ and $0 < s < 2$. If $\mu \in M^1_{p,s}(\Delta^*)$, then $\Phi(\mu) \in N(p,s)$.*

Proof. Let $\mu \in M^1_{p,s}(\Delta^*)$ and f_μ be the normalized quasiconformal mapping f_μ of $\mathbf{C}$ whose complex dilatation is μ in Δ^* and is zero in Δ. Set $\widehat{f}(\zeta) = s \circ f_\mu \circ s(\zeta)$, where $s(\zeta) = 1/\zeta$. Then $\widehat{f}$ is a quasiconformal mapping of the whole plane $\mathbf{C}$ whose complex dilatation $\mu_{\widehat{f}}(\zeta)$ satisfies $|\mu_{\widehat{f}}(\zeta)| = |\mu(\frac{1}{\zeta})|$ in Δ and is zero in Δ^*. By a change of variable, we conclude that

$$\lambda_{\mu_{\widehat{f}}} = \frac{|\mu_{\widehat{f}}(\zeta)|^p}{(1-|\zeta|^2)^{2-s}}\,d\xi\,d\eta \in CM_s(\Delta) \quad \text{and} \quad \|\lambda_{\mu_{\widehat{f}}}\|_{C,s} = \|\lambda_\mu\|_{C,s}. \tag{22}$$

Let $g = \gamma_{\widehat{f}} \circ \widehat{f} \circ \beta_z$, where $\gamma_{\widehat{f}}$ and β_z are defined in (20). The area theorem of univalent functions yields

$$\iint_\Delta J_g(\zeta)\,d\xi\,d\eta \le \pi, \tag{23}$$

where J_g is the Jacobian determinant of g. Noting that $\mu_g = \mu_{\widehat{f}\circ\beta_z}$ and $\|\mu_g\|_\infty = ||\mu||_\infty$, by (21), (23) and Hölder's inequality, we get

$$\begin{aligned} |S_{\widehat{f}}(z)|^p(|z|^2-1)^{2p} &= \left(\frac{3}{2\pi}\right)^p \left|\iint_\Delta (\mu_{\widehat{f}\circ\beta_z}\partial g)(\zeta)\,d\xi\,d\eta\right|^p \\ &\le \left(\frac{3}{2\pi}\right)^p \frac{\pi}{(1-\|\mu_g\|^2_\infty)^{\frac{p}{2}}} \iint_\Delta |\,\mu_{\widehat{f}\circ\beta_z}(\zeta)\,|^p\ d\xi\,d\eta \\ &= C_1(||\mu||_\infty) \iint_\Delta \frac{|\mu_{\widehat{f}}(w)|^p(|z|^2-1)^2}{|\,w-z\,|^4}\,du\,dv, \end{aligned} \tag{24}$$

where $C_1(||\mu||_\infty) = \left(\frac{3}{2}\right)^p \frac{\pi^{1-p}}{(1-\|\mu\|_\infty^2)^{\frac{p}{2}}}$.

Consequently, for $b \in \Delta^*$, set $a = 1/b \in \Delta$, by (22), (24) and Lemma 2.2, we have

$$\begin{aligned}
&\iint_{\Delta^*} |S_{\widehat{f}}(z)|^p(|z|^2-1)^{2p+s-2}\frac{(|b|^2-1)^s}{|1-\overline{b}z|^{2s}}\,dx\,dy \\
&\le C_1(||\mu||_\infty)\iint_\Delta \frac{|\mu_{\widehat{f}}(w)|^p(1-|a|^2)^s}{(1-|w|^2)^{2-s}}\,du\,dv \iint_\Delta \frac{(1-|w|^2)^{2-s}(1-|z|^2)^s}{|1-wz|^4|1-\overline{a}z|^{2s}}\,dx\,dy \\
&\le C_2(||\mu||_\infty)\iint_\Delta \frac{|\mu_{\widehat{f}}(w)|^p}{(1-|w|^2)^{2-s}}\frac{(1-|a|^2)^s}{|1-\overline{a}w|^{2s}}\,du\,dv \\
&\le C_3(||\mu||_\infty)\|\lambda_{\mu_{\widehat{f}}}\|_{C,s} = C_3(||\mu||_\infty)\|\lambda_\mu\|_{C,s}.
\end{aligned} \tag{25}$$

Noting that $|S_{\widehat{f}}(\zeta)| = |S_{f_\mu}(\frac{1}{\zeta})|\frac{1}{|\zeta|^4}$, we get from (25) that

$$\|S_{f_\mu}\|^p_{N_{p,s}} = \sup_{a\in\Delta^*}\iint_{\Delta^*} |S_{\widehat{f}}(z)|^p(|z|^2-1)^{2p+s-2}\frac{(|a|^2-1)^s}{|1-\overline{a}z|^{2s}}\,dx\,dy < \infty. \tag{26}$$

Which implies that $\Phi(\mu) = S_{f_\mu} \in N(p,s)$ if $\mu \in M^1_{p,s}(\Delta^*)$. The proof follows. □

Before proving Theorem 1.2, we first recall some basic facts about the infinite dimensional holomorphy (see [11, p. 206], [12, p. 86–87]). Let E and F be two complex Banach space and U be an open subset in E, a mapping $f\colon U \to F$ is holomorphic if and only if it is continous (locally boundedness is also enough) and the complex Gateaux derivative $d_x(\lambda)$ defined as

$$d_x(\lambda) = \lim_{t\to 0}\frac{f(x+t\lambda)-f(x)}{t}$$

exists for each $(x,\lambda) \in U \times E$.

Let F^* denote the dual space of F in the usual sense. For a subset A of F^*, we define $A^\perp = \{y \in F\colon y^*(y) = 0, y^* \in A\}$. A subset A is called total if $A^\perp = \{0\}$.

Proposition 2.5. [11, 12] *$f\colon U \to F$ is holomorphic if and only if it satisfies one of the following conditions.*

(i) *The mapping f is local bounded and for every $(x,\lambda) \in U \times E$, the mapping $t \mapsto f(x+t\lambda)$ is holomorphic from an open neighborhood of zero in the complex plane $\mathbf{C}$ to F.*
(ii) *The mapping f is continuous and there exists a total subset A of F^* such that for every $y^* \in A$, the function $y^*(f)\colon U \to \mathbf{C}$ is holomorphic.*

We are now in a position to prove Theorem 1.2. Our proof is based on the proof of Theorem 3 in Cui [5].

Proof of Theorem 1.2.. We first show that mapping $\Phi\colon M^1_{p,s}(\Delta^*) \to N(p,s)$ is continuous. Let $\widehat{\mu} \in M^1_{p,s}(\Delta^*)$, $\widehat{\nu} \in M^1_{p,s}(\Delta^*)$. It is sufficient to show that there is a constant $C(\|\widehat{\mu}\|_\infty, \|\widehat{\nu}\|_\infty)$ such that

$$\|\Phi(\widehat{\mu}) - \Phi(\widehat{\nu})\|_{N_{p,s}} \le C(\|\widehat{\mu}\|_\infty, \|\widehat{\nu}\|_\infty)\|\widehat{\mu}-\widehat{\nu}\|_s. \tag{27}$$

Set $\widehat{f_1} = f_{\widehat{\mu}}$, $\widehat{f_2} = f_{\widehat{\nu}}$ and $f_i(\zeta) = s \circ \widehat{f_i} \circ s(\zeta)$, $i = 1,2$, where $s(\zeta) = 1/\zeta$. Then f_1 is a quasiconformal mapping of $\mathbf{C}$ whose complex dilatation is equal to $\mu(\zeta) = \widehat{\mu}(s(\zeta))\frac{\overline{s'(\zeta)}}{s'(\zeta)}$ in Δ and is zero in Δ^*, while f_2 is a quasiconformal mapping of $\mathbf{C}$ whose complex dilatation is equal to $\nu(\zeta) = \widehat{\nu}(s(\zeta))\frac{\overline{s'(\zeta)}}{s'(\zeta)}$ in Δ and is zero in Δ^*. Thus the correspondence between μ and $\widehat{\mu}$ is one-to-one and $\|\widehat{\mu}\|_\infty = \|\mu\|_\infty$, $\|\widehat{\nu}\|_\infty = \|\nu\|_\infty$. By a change of variable, we conclude that

$$\|\lambda_{\mu_{\hat f}}\|_{C,s} = \|\lambda_\mu\|_{C,s}\,, \quad \|\lambda_{\nu_{\hat f}}\|_{C,s} = \|\lambda_\nu\|_{C,s} \tag{28}$$

and

$$\|\Phi(\widehat{\mu}) - \Phi(\widehat{\nu})\|^p_{N_{p,s}} = \sup_{b\in\Delta^*} \iint_{\Delta^*} |S_{f_1}(z) - S_{f_2}(z)|^p (|z|^2-1)^{2p+s-2} \frac{(|b|^2-1)^s}{|1-\overline{b}z|^{2s}}\, dx\, dy. \tag{29}$$

Let $g^\mu = \gamma_{f_1} \circ f_1 \circ \beta_z$ and $g^\nu = \gamma_{f_2} \circ f_2 \circ \beta_z$. By Lemma 2.3, we have

$$S_{f_1}(z) - S_{f_2}(z) = -\frac{3(|z|^2-1)^{-2}}{2\pi} \iint_\Delta \overline{\partial}(g^\mu - g^\nu)(w)\, du\, dv. \tag{30}$$

Set $\mu_{\beta_z}(w) = \mu(\beta_z(w))\frac{\overline{\beta_z'(w)}}{\beta_z'(w)}$, $\nu_{\beta_z}(w) = \nu(\beta_z(w))\frac{\overline{\beta_z'(w)}}{\beta_z'(w)}$. Let H be the Beuring–Ahlfors operator defined as

$$H(\phi)(\zeta) = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\phi(z)}{(\zeta - z)^2}\, dx\, dy,$$

the integral is understood in the sense of Cauchy principal value. The representation theorem of quasiconformal mapping says that $\overline{\partial} g^\mu = \mu_{\beta_z}(I + H\overline{\partial} g^\mu)$, $\overline{\partial} g^\nu = \nu_{\beta_z}(I + H\overline{\partial} g^\nu)$ (see [1, Chapter V]). Consequently, we conclude that

$$\begin{aligned}\overline{\partial}(g^\mu - g^\nu) &= \mu_{\beta_z}(I + H\overline{\partial} g^\mu) - \nu_{\beta_z}(I + H\overline{\partial} g^\nu)\\ &= \mu_{\beta_z} - \nu_{\beta_z} + \mu_{\beta_z} H\overline{\partial} g^\mu - \mu_{\beta_z} H\overline{\partial} g^\nu + \mu_{\beta_z} H\overline{\partial} g^\nu - \nu_{\beta_z} H\overline{\partial} g^\nu\\ &= (\mu_{\beta_z} - \nu_{\beta_z})(H\overline{\partial} g^\nu + I) + \mu_{\beta_z} H\overline{\partial}(g^\mu - g^\nu).\end{aligned}$$

Since $\partial g^\nu = I + H\overline{\partial} g^\nu$, we have

$$\overline{\partial}(g^\mu - g^\nu) = (I - \mu_{\beta_z} H)^{-1}(\mu_{\beta_z} - \nu_{\beta_z})(I + H\overline{\partial} g^\nu) = (I - \mu_{\beta_z} H)^{-1}((\mu_{\beta_z} - \nu_{\beta_z})\partial g^\nu).$$

Thus it follows from (30) that

$$S_{f_2}(z) - S_{f_1}(z) = -\frac{3(|z|^2-1)^{-2}}{2\pi} \iint_\Delta (I - \mu_{\beta_z} H)^{-1}((\mu_{\beta_z} - \nu_{\beta_z})\partial g^\nu)(w)\, du\, dv. \tag{31}$$

Since $(I - \mu_{\beta_z} H)^{-1} = I + \mu_{\beta_z} H (I - \mu_{\beta_z} H)^{-1}$, we have

$$\begin{aligned}&S_{f_2}(z) - S_{f_1}(z)\\ &= -\frac{3(|z|^2-1)^{-2}}{2\pi} \iint_\Delta (I - \mu_{\beta_z} H)^{-1}((\mu_{\beta_z} - \nu_{\beta_z})\partial g^\nu)(\zeta)\, d\xi\, d\eta\\ &= -\frac{3(|z|^2-1)^{-2}}{2\pi} \iint_\Delta ((\mu_{\beta_z} - \nu_{\beta_z})\partial g^\nu)(\zeta)\, d\xi\, d\eta\\ &\quad - \frac{3(|z|^2-1)^{-2}}{2\pi} \iint_\Delta \mu_{\beta_z} H (I - \mu_{\beta_z} H)^{-1}((\mu_{\beta_z} - \nu_{\beta_z})\partial g^\nu)(\zeta)\, d\xi\, d\eta\\ &= L_1 + L_2,\end{aligned} \tag{32}$$

where

$$L_1 = -\frac{3(|z|^2-1)^{-2}}{2\pi} \iint_\Delta ((\mu_{\beta_z} - \nu_{\beta_z})\partial g^\nu)(\zeta)\, d\xi\, d\eta$$

and

$$L_2 = -\frac{3(|z|^2-1)^{-2}}{2\pi} \iint_\Delta \mu_{\beta_z} H (I - \mu_{\beta_z} H)^{-1}((\mu_{\beta_z} - \nu_{\beta_z})\partial g^\nu)(\zeta)\, d\xi\, d\eta.$$

By using the method similar to (24), we get

$$|L_1|^p(|z|^2-1)^{2p} \le C(\|\mu\|_\infty)\iint_\Delta \frac{|\mu(w)-\nu(w)|^p(|z|^2-1)^2}{|\ w-z\ |^4}\,du\,dv. \tag{33}$$

Consequently, similar to (25), by Lemma 2.2 and a change of variable, we obtain

$$\begin{aligned}&\sup_{b\in\Delta^*}\iint_{\Delta^*}|L_1(z)|^p(|z|^2-1)^{2p+s-2}\frac{(|b|^2-1)^s}{|1-\overline{b}z|^{2s}}\,dx\,dy\\ &\le C(\|\mu\|_\infty)\sup_{a\in\Delta}\iint_\Delta \frac{|\mu(w)-\nu(w)|^p}{(1-|w|^2)^{2-s}}\frac{(1-|a|^2)^s}{|1-\overline{a}w|^{2s}}\,du\,dv.\end{aligned} \tag{34}$$

We now estimate L_2. It is noted that when $\|\mu\|_\infty<1$, the operator $I-\mu H$ is invertible on $L^2(\Delta)$ and the norm of its inverse $(I-\mu H)^{-1}$ is less than $1/(1-\|\mu\|_\infty)$. Thus we have

$$\begin{aligned}|\ L_2|^2(|z|^2-1)^4 &= \left(\frac{3}{2\pi}\right)^2\left|\iint_\Delta \mu_{\beta_z}H(I-\mu_{\beta_z}H)^{-1}((\mu_{\beta_z}-\nu_{\beta_z})\partial g^\nu)(\zeta)\,d\xi\,d\eta\right|^2\\ &\le \frac{9}{4\pi^2(1-\|\mu\|_\infty)^2}\iint_\Delta|\mu_{\beta_z}|^2\,d\xi\,d\eta\iint_\Delta|((\mu_{\beta_z}-\nu_{\beta_z})\partial g^\nu)(\zeta)|^2\,d\xi\,d\eta\\ &\le \frac{9\|\mu-\nu\|_\infty^2}{4\pi(1-\|\mu\|_\infty)^2(1-\|\nu\|_\infty^2)}\iint_\Delta|\mu_{\beta_z}|^2\,d\xi\,d\eta\\ &= \frac{9\|\mu-\nu\|_\infty^2}{4\pi(1-\|\mu\|_\infty)^2(1-\|\nu\|_\infty^2)}\iint_\Delta\frac{|\mu(\zeta)|^2(|z|^2-1)^2}{|\zeta-z|^4}\,d\xi\,d\eta.\end{aligned} \tag{35}$$

Noting that $p\ge 2$, by Hölder's inequality, we get

$$\begin{aligned}|L_2|^p &\le \left(\frac{9\|\mu-\nu\|_\infty^2(|z|^2-1)^{-2}}{4\pi(1-\|\mu\|_\infty)^2(1-\|\nu\|_\infty^2)}\right)^{\frac{p}{2}}\iint_\Delta\frac{|\mu(\zeta)|^p}{|\zeta-z|^4}\,d\xi\,d\eta\left(\iint_\Delta\frac{1}{|\zeta-z|^4}\,d\xi\,d\eta\right)^{\frac{p}{2}-1}\\ &\le C_1(\|\mu\|_\infty,\|\nu\|_\infty)\|\mu-\nu\|_\infty^p\iint_\Delta\frac{|\mu(\zeta)|^p}{|\zeta-z|^4(|z|^2-1)^{2p-2}}\,d\xi\,d\eta,\end{aligned} \tag{36}$$

Similar to L_1, we can deduce that

$$\begin{aligned}&\sup_{b\in\Delta^*}\iint_{\Delta^*}|L_2(z)|^p(|z|^2-1)^{2p+s-2}\frac{(|b|^2-1)^s}{|1-\overline{b}z|^{2s}}\,dx\,dy\\ &\le C_2(\|\mu\|_\infty,\|\nu\|_\infty)\|\mu-\nu\|_\infty^p\sup_{a\in\Delta}\iint_\Delta\frac{|\mu(w)|^p}{(1-|w|^2)^{2-s}}\frac{(1-|a|^2)^s}{|1-\overline{a}w|^{2s}}\,du\,dv.\end{aligned} \tag{37}$$

Combining (28), (29), (32), (34) and (37), we deduce that (27) holds and thus the mapping $\Phi\colon M^1_{p,s}(\Delta^*)\to N(p,s)$ is continuous.

We now prove that the Bers projection $\Phi\colon M^1_{p,s}(\Delta^*)\to N(p,s)$ is holomorphic. For each $z\in\Delta$, we define a continuous linear functional l_z on the Banach space $N(p,s)$ by $l_z(\varphi)=\varphi(z)$ for $\varphi\in N(p,s)$. Then the set $A=\{l_z\colon z\in\Delta\}$ is a total subset of the dual space of $N(p,s)$. Now for each $z\in\Delta$, each pair $(\mu,\nu)\in M^1_{p,s}(\Delta^*)\times M_{p,s}(\Delta^*)$ and small t in the complex plane, by the well known holomorphic dependence of quasiconformal mappings on parameters (see [11, Theorem 3.1 in Chapter II], [1, Chapter V]), we conclude that $l_z(\Phi(\mu+t\nu))=S_{f_{\mu+t\nu}}(z)$ is a holomorphic function of t. From Proposition 2.5, the Bers projection $\Phi\colon M^1_{p,s}(\Delta^*)\to N(p,s)$ is holomorphic. This completes the proof. □

Checking the proof of Theorem 1.2, we can show the following

Theorem 2.6. *Let $p \geq 2$ and $0 < s < 2$. The Bers projection $\Phi\colon M^1_{p,s,0}(\Delta^*) \to N_0(p,s)$ is holomorphic.*

We now prove Theorem 1.3.

Proof of Theorem 1.3. It follows from Theorem A and Proposition 2.4 that the mapping $L_{z_0}\colon M^1_{p,s}(\Delta^*) \to F(p,s)^0$ is well defined. We can prove $L_{z_0}\colon M^1_{p,s}(\Delta^*) \to F(p,s)^0$ is holomorphic by the same reasoning as the proof of the holomorphy of $\Phi\colon M^1_{p,s}(\Delta^*) \to N(p,s)$. Thus it is enough to show that $L_{z_0}\colon M^1_{p,s}(\Delta^*) \to F(p,s)^0$ is continuous. For $\mu, \nu \in M^1_{p,s}(\Delta^*)$, it follows from the proof of [11, Theorem 3.1 in Chapter II] that

$$\sup_{z\in\Delta}(1-|z|^2)\left|\frac{g''_\mu}{g'_\mu}-\frac{g''_\nu}{g'_\nu}\right| \leq C(\|\mu\|_\infty)\|\mu-\nu\|_\infty. \tag{38}$$

By Theorem 1.2, we conclude that

$$\|S_{g_\mu}(z)-S_{g_\nu}(z)\|_{N_{p,s}} \leq C_1(\|\mu\|_\infty,\|\nu\|_\infty)\|\mu-\nu\|_s. \tag{39}$$

It follows from Chapter 4 in [14] that there is a constant $C_2 > 0$ which is independent of μ and ν such that

$$\begin{aligned}&\iint_\Delta \left|\frac{g''_\mu}{g'_\mu}-\frac{g''_\nu}{g'_\nu}\right|^p \frac{(1-|z|^2)^{p-2}(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\,dx\,dy\\ &\leq C_2\left|\frac{g''_\mu}{g'_\mu}(0)-\frac{g''_\nu}{g'_\nu}(0)\right|^p + C_2\iint_\Delta\left|\left(\frac{g''_\mu}{g'_\mu}\right)'-\left(\frac{g''_\nu}{g'_\nu}\right)'\right|^p\frac{(1-|z|^2)^{2p-2}(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\,dx\,dy.\end{aligned} \tag{40}$$

By the definition of the Schwarzian derivative, we get

$$\begin{aligned}\left|\left(\frac{g''_\mu}{g'_\mu}\right)'-\left(\frac{g''_\nu}{g'_\nu}\right)'\right|^p &\leq 2^p|S_{g_\mu}-S_{g_\nu}|^p + 2^p\left|\left(\frac{g''_\mu}{g'_\mu}\right)^2-\left(\frac{g''_\nu}{g'_\nu}\right)^2\right|^p\\ &= 2^p|S_{g_\mu}-S_{g_\nu}|^p + 2^p\left|\frac{g''_\mu}{g'_\mu}+\frac{g''_\nu}{g'_\nu}\right|^p\left|\frac{g''_\mu}{g'_\mu}-\frac{g''_\nu}{g'_\nu}\right|^p.\end{aligned} \tag{41}$$

Taking $z = 0$ in (38), we get

$$\left|\frac{g''_\mu}{g'_\mu}(0)-\frac{g''_\nu}{g'_\nu}(0)\right|^p \leq C^p(\|\mu\|_\infty)\|\mu-\nu\|^p_\infty. \tag{42}$$

It follows from (38), (39) and (41) that

$$\begin{aligned}&\iint_\Delta\left|\left(\frac{g''_\mu}{g'_\mu}\right)'-\left(\frac{g''_\nu}{g'_\nu}\right)'\right|^p\frac{(1-|z|^2)^{2p-2}(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\,dx\,dy\\ &\leq 2^p\iint_\Delta |S_{g_\mu}-S_{g_\nu}|^p\frac{(1-|z|^2)^{2p-2}(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\,dx\,dy\\ &\quad + 2^p\iint_\Delta\left|\frac{g''_\mu}{g'_\mu}+\frac{g''_\nu}{g'_\nu}\right|^p\left|\frac{g''_\mu}{g'_\mu}-\frac{g''_\nu}{g'_\nu}\right|^p\frac{(1-|z|^2)^{2p-2}(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\,dx\,dy\\ &\leq 2^pC_1^p(\|\mu\|_\infty,\|\nu\|_\infty)\|\mu-\nu\|^p_s\\ &\quad + 2^p\sup_{z\in\Delta}(1-|z|^2)^p\left|\frac{g''_\mu}{g'_\mu}-\frac{g''_\nu}{g'_\nu}\right|^p\iint_\Delta\left|\frac{g''_\mu}{g'_\mu}+\frac{g''_\nu}{g'_\nu}\right|^p\frac{(1-|z|^2)^{p-2}(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\,dx\,dy\\ &\leq 2^pC_1^p(\|\mu\|_\infty,\|\nu\|_\infty)\|\mu-\nu\|^p_s\\ &\quad + 4^pC^p(\|\mu\|_\infty)(\|\log g'_\mu\|_{F(p,p-2,s)}+\|\log g'_\nu\|_{F(p,p-2,s)})^p\|\mu-\nu\|^p_\infty.\end{aligned} \tag{43}$$

Combining (40), (42) and (43), we get

$$\|L_{z_0}(\mu)-L_{z_0}(\nu)\|_{F(p,p-2,s)}$$
$$\leq C_3\left(\|\mu\|_\infty,\|\nu\|_\infty,\|\log g'_\mu\|_{F(p,p-2,s)},\|\log g'_\nu\|_{F(p,p-2,s)}\right)\|\mu-\nu\|_s.$$

This completes the proof of Theorem 1.3. □

Similarly, we have the following

Theorem 2.7. *Let $p\geq 2$ and $0<s<2$. For $z_0\in\Delta^*$, the pre-Bers projection mapping $L_{z_0}\colon M^1_{p,s,0}(\Delta^*)\to F_0(p,s)^0$ is holomorphic.*

3. Proofs of Theorem 1.1 and Theorem 1.4

In this section, we prove Theorem 1.1 and Theorem 1.4.

Proof of Theorem 1.1. Let $\log f'\in\widetilde{T}_{F(p,s)}$. Then f is a quasiconformal mapping of the complex plane $\mathbf{C}$ whose complex dilatation μ satisfies $\lambda_\mu=\frac{|\mu(z)|^p}{(|z|^2-1)^{2-s}}\,dx\,dy\in CM_s(\Delta^*)$ and equals to zero in Δ. Let f^t be the quasiconformal mapping in $\mathbf{C}$ with $f^{-1}(\infty)=(f^t)^{-1}(\infty)$ and $\overline{\partial}f^t=t\mu\partial f^t$. Consider the path $t\longmapsto\log(f^t)',0\leq t\leq 1$, in the space $F(p,s)$. Set $g=f^t$ and $h=f^s$. By Theorem 1.3, we conclude that

$$\|\log g'-\log h'\|_{F(p,p-2,s)}\leq C\left(\|\mu\|_s,\|\log g'\|_{F(p,p-2,s)},\|\log h'\|_{F(p,p-2,s)}\right)|t-s|.$$

This implies that the path $t\longmapsto\log(f^t)',0\leq t\leq 1$, is continuous in the space $F(p,s)$. Consequently, each $\log f'\in\widetilde{T}_{F(p,s)}$ can be connected by a continuous path to an element $\log\varphi'\in F(p,s)$, where φ is a Möbius transformation of $\mathbf{C}$. If $\varphi(\Delta)$ is unbounded, then $f(\zeta)=\varphi(\zeta)=\infty$ for some $\zeta\in\partial\Delta$. Otherwise $\varphi(\Delta)$ is bounded, we consider the path $r\longmapsto\log\varphi_r'$, where $\varphi_r=\varphi(rz)$, $0\leq r\leq 1$. It is easy to see that this is a path which connects the point $\log\varphi'$ to the point 0 in $F(p,s)$. It turns out that $\widetilde{T}_b=\{\log f'\in\widetilde{T}_{F(p,s)}\colon f(\Delta)\text{ bounded }\}$ and $\widetilde{T}_\theta=\{\log f'\in\widetilde{T}_{F(p,s)}\colon \lim_{z\to e^{i\theta}}f(z)=\infty\}$, $0\leq\theta\leq 2\pi$, are connected. By [24], elements in different classes can not be connected even in Bloch space. We conclude that $\widetilde{T}_b$ and $\widetilde{T}_\theta$ are the connected components of $\widetilde{T}_{F(p,s)}$. □

Before proving Theorem 1.4, we need a lemma which was proved by Shen and Wei in [16].

Lemma 3.1. *Let h be a quasisymmetric homeomorphism on S^1 with normalized decomposition $h=f^{-1}\circ g$ and ν be the complex dilatation of a quasiconformal extension of h^{-1} to Δ. Then*

$$\frac{1}{36}(1-|z|^2)^2|S_f(z)|^2\leq U^2(f,z)\leq\phi_h^2(\overline{z})\leq\frac{1}{\pi}\iint_\Delta\frac{|\nu(w)|^2}{1-|\nu(w)|^2}\frac{1}{|1-\overline{z}w|^4}\,du\,dv. \tag{44}$$

Proof of Theorem 1.4. It follows from Theorem A that (1) $\Longleftrightarrow$ (2). From Theorem 1.3, we conclude that (1) $\Longrightarrow$ (3). Lemma 3.1 gives (4) $\Longleftrightarrow$ (5) and (4) $\Longrightarrow$ (2). Thus it remains to show that (3) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (5).

We first prove that (3) $\Longrightarrow$ (5). Let h be a quasisymmetric homeomorphism on the unit circle S^1. Then there exists a unique pair of conformal mappings $f\in S_Q$ on Δ and g on Δ^*, such that $f(0)=f'(0)-1=0$, $g(\infty)=\infty$ and $h=f^{-1}\circ g$ on S^1 (see [11, Lemma 1.1 in Chapter III]). Suppose f can be extended to a quasiconformal mapping of the whole plane $\mathbf{C}$, which is also denoted by f, such that its complex dilatation μ satisfies $\frac{|\mu(z)|^p}{(|z|^2-1)^{2-s}}\,dx\,dy\in CM_{s,0}(\Delta^*)$. It is noted that $\widehat{H}=g^{-1}\circ f$ is a quasiconformal extension of h^{-1} to Δ^* and has the same complex dilatation μ as f.

Then $H = j \circ \widehat{H} \circ j$, where $j(z) = 1/\overline{z}$, is a quasiconformal extension of h^{-1} to Δ with complex dilatation $\nu(z)$ satisfying $|\nu(z)| = |\mu(1/\overline{z})|$. A computation shows that $\frac{|\nu(z)|^p}{(1-|z|^2)^{2-s}}\,dx\,dy \in CM_{s,0}(\Delta)$. By Lemma 2.2 and Lemma 3.1, we conclude that (5) holds.

We now show that (1) $\Longrightarrow$ (3). Suppose that (1) holds. Noting that $F_0(p,s)$ is a subspace of the little Bloch space B_0^1, we have

$$\lim_{|z|\to 1}(1-|z|^2)|f''(z)/f'(z)| = 0.$$

Becker and Pommerenke (see [4]) constructed a quasiconformal extension of the conformal mapping f to the whole plane $\mathbf{C}$ by the following formula

$$f(z) = f(1/\overline{z}) + f'(1/\overline{z})(z - 1/\overline{z}), \quad z \in \Delta^*.$$

By some computations we have

$$|\mu(z)| = |1/z|^2(1-|1/z|^2)|f''(1/\overline{z})/f'(1/\overline{z})|.$$

For $z, b \in \Delta^*$, we set $w = 1/\overline{z}$ and $b = 1/\overline{a}$. A change of variable gives

$$\begin{aligned}&\iint_{\Delta^*}\frac{|\mu(z)|^p}{(|z|^2-1)^{2-s}}\frac{(|b|^2-1)^s}{|1-\overline{b}z|^{2s}}\,dx\,dy\\ &\le \iint_{\Delta}|f''(w)/f'(w)|^p(1-|w|^2)^{p-2+s}\frac{(1-|a|^2)^s}{|1-\overline{a}w|^{2s}}\,du\,dv.\end{aligned} \tag{45}$$

Noting that $\log f' \in F_0(p,s)$ and $|b| \to 1$ if and only if $|a| \to 1$, we conclude that

$$\frac{|\mu(z)|^p}{(|z|^2-1)^{2-s}}\,dx\,dy \in CM_{s,0}(\Delta^*).$$

The proof follows. □

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