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ON FALCONER'S FORMULA FOR THE GENERALIZED RÉNYI DIMENSION OF A SELF-AFFINE MEASURE

Ian D. Morris

University of Surrey, Department of Mathematics
Guildford GU2 7XH, U.K.; i.morris@surrey.ac.uk

Abstract. We investigate a formula of Falconer which describes the typical value of the generalised Rényi dimension, or generalised q-dimension, of a self-affine measure in terms of the linear components of the affinities. We show that in contrast to a related formula for the Hausdorff dimension of a typical self-affine set, the value of the generalised q-dimension predicted by Falconer's formula varies discontinuously as the linear parts of the affinities are changed. Conditionally on a conjecture of Bochi and Fayad, we show that the value predicted by this formula for pairs of two-dimensional affine transformations is discontinuous on a set of positive Lebesgue measure. These discontinuities derive from discontinuities of the lower spectral radius which were previously observed by the author and Bochi.

1. Introduction

If $\mathsf{T} := (T_1, \ldots, T_N)$ is a finite collection of transformations of a complete metric space X, and each T_i is a contraction in the sense that for some $\lambda < 1$ one has $d(T_i x, T_i y) \leq \lambda d(x, y)$ for all $x, y \in X$, it is well-known that there exists a unique nonempty compact set $Z_{\mathsf{T}} \subseteq X$ such that

$$Z_{\mathsf{T}} = \bigcup_{i=1}^{N} T_i \left(Z_{\mathsf{T}} \right),$$

see for example [11, 18]. The case in which $X = \mathbf{R}^d$ and the transformations T_i are affine—in which case Z_{T} is called a *self-affine set*—has been the subject of intense study over the last few decades (for a recent survey we note [13]).

Various notions of fractal dimension have been investigated for both general and special classes of self-affine set. In certain special cases where the linear parts of the affinities preserve or permute the horizontal and vertical axes of $\mathbf{R}^2$, explicit formulæ for the Hausdorff dimension and box dimension exist (see e.g. [3, 5, 16, 21, 23]). In the context of general self-affine sets, a landmark result of Falconer [8] established the Hausdorff and box dimensions of "typical" self-affine sets in a sense which we now describe. Let $M_d(\mathbf{R})$ denote the set of all $d \times d$ real matrices, and recall that for $A \in M_d(\mathbf{R})$ we define the *singular values* of A, denoted $\sigma_1(A), \ldots, \sigma_d(A)$, to be the non-negative square roots of the eigenvalues of the positive semi-definite matrix A^*A, listed in decreasing order. Let us define a function $\varphi\colon (0, +\infty) \times M_d(\mathbf{R}) \to [0, +\infty)$ by

$$\varphi^s(A) := \begin{cases} \sigma_1(A) \cdots \sigma_k(A) \sigma_{k+1}(A)^{s-k}, & k \leq s \leq k+1 \leq d, \\ |\det A|^{\frac{s}{d}}, & s \geq d, \end{cases}$$

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and for every $A_1, \ldots, A_N$ in the open unit ball of $M_d(\mathbf{R})$ define the *affinity dimension* or *singularity dimension* of $(A_1, \ldots, A_N)$ to be the quantity

$$\mathfrak{s}(A_1, \ldots, A_N) := \inf \left\{ s > 0 \colon \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{N} \varphi^s \left(A_{i_1} \cdots A_{i_n} \right) < \infty \right\}.$$

Falconer showed that for any fixed invertible $d \times d$ matrices $A_1, \ldots, A_N$ with Euclidean norm strictly less than $\frac{1}{3}$, for Lebesgue almost all $v_1, \ldots, v_N \in \mathbf{R}^d$ the self-affine set Z_{T} associated to the collection of affine transformations $\mathsf{T} := (T_1, \ldots, T_N)$ defined by $T_i x := A_i x + v_i$ satisfies

$$\dim_H (Z_{\mathsf{T}}) = \dim_B (Z_{\mathsf{T}}) = \min \{ \mathfrak{s}(A_1, \ldots, A_N), d \},$$

see [8]; the bound on the norm was subsequently relaxed to $\frac{1}{2}$ by B. Solomyak [25], and to 1 by Jordan, Simon and Pollicott for a notion of "almost self-affine set" which incorporates additional random translations [19]. While it is well-known that the Hausdorff dimension of Z_{T} can fail to depend continuously on the affinites $T_1, \ldots, T_N$ (see e.g. [11, Example 9.10]), it was recently shown by Feng and Shmerkin in [15] that the affinity dimension $\mathfrak{s}$ is a continuous function of $(A_1, \ldots, A_N)$. An alternative proof of this statement was subsequently given by the author [24].

In this article we will focus not on self-affine sets but on self-affine *measures*. A Borel probability measure μ on $\mathbf{R}^d$ is called self-affine if there exist a probability vector $\mathsf{p} = (p_1, \ldots, p_N)$ and a collection of affinities $\mathsf{T} = (T_1, \ldots, T_N)$ such that

$$\mu(A) = \sum_{i=1}^{N} p_i \mu \left(T_i^{-1} A \right)$$

for every Borel set $A \subseteq \mathbf{R}^d$. If $T_1, \ldots, T_N$ are contractions then for each probability vector p with all probabilities nonzero there exists a unique Borel probability measure $\mu_{\mathsf{p},\mathsf{T}}$ satisfying the above functional equation (see e.g. [9, Theorem 2.8]), and the support of that measure is equal to the associated self-affine set Z_{T}. In this article our interest is in the *generalised q-dimension* or *generalised Rényi dimension* of a self-affine measure, which is defined as follows. For each $r > 0$ let $\mathcal{M}_r$ denote the set of all r-mesh cubes on $\mathbf{R}^d$, that is, the set of all d-dimensional cubes of the form $[j_1 r, (j_1+1)r) \times [j_2 r, (j_2+1)r) \times \cdots \times [j_d r, (j_d+1)r)$ where $j_1, \ldots, j_d \in \mathbf{Z}$. For $q > 1$ we define

$$M_r(q, \mu) := \sum_{C \in \mathcal{M}_r} \mu(C)^q$$

for every $r > 0$, and

$$\underline{D}_q(\mu) := \liminf_{r \to 0} \frac{\log M_r(q, \mu)}{(q-1) \log r}, \quad \overline{D}_q(\mu) := \limsup_{r \to 0} \frac{\log M_r(q, \mu)}{(q-1) \log r}.$$

If $\underline{D}_q(\mu)$ and $\overline{D}_q(\mu)$ are equal then we define the generalised q-dimension of μ to be their common value and denote this by $D_q(\mu)$. For $q > 1$ the generalised q-dimension admits an alternative expression as a limit of certain integrals [22]. In [10, Theorem 6.2], Falconer characterised the generalised Rényi dimensions of typical self-affine measures in a similar manner to his earlier characterisation of the Hausdorff and box dimensions of typical self-affine sets:

Theorem 1.1. (Falconer) *Let $(A_1, \ldots, A_N)$ be invertible linear transformations of $\mathbf{R}^d$ such that $\|A_i\| < \frac{1}{2}$ for every i, let $\mathsf{p} = (p_1, \ldots, p_N)$ be a probability vector with* *all entries nonzero, and for each $q > 1$ define $\mathfrak{r}_q(A_1, \ldots, A_N, \mathsf{p})$ to be the quantity*

$$\sup\left\{s > 0\colon \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{N} \varphi^s(A_{i_1}\cdots A_{i_n})^{1-q} p_{i_1}^q \cdots p_{i_n}^q < \infty\right\}.$$

If $1 < q \leq 2$ then for Lebesgue-almost-every $(v_1, \ldots, v_N) \in \mathbf{R}^{Nd}$ the self-affine measure $\mu_{\mathsf{p},\mathsf{T}}$ corresponding to the transformations $T_i x := A_i x + v_i$ and the probability vector p satisfies $D_q(\mu_{\mathsf{p},\mathsf{T}}) = \min\{\mathfrak{r}_q(A_1, \ldots, A_N, \mathsf{p}), d\}$.

In the later article [12] this result was extended to a more general class of almost self-affine measures, under the weaker hypotheses $\max \|A_i\| < 1$ and $q > 1$ but requiring randomised translations in a similar manner to [19]. An alternative extension of this result, which allows $q \in [1, Q]$ for certain values of $Q > 2$, was recently obtained by Barral and Feng [4, §6]. In view of the recent work of Feng and Shmerkin [15] on the continuity of the formula $\mathfrak{s}(A_1, \ldots, A_N)$ for the typical dimension of a self-affine set, it is natural to ask whether the formula $\mathfrak{r}_q(A_1, \ldots, A_N, \mathsf{p})$ for the typical dimension of a self-affine measure is also continuous with respect to changes in the matrices $A_1, \ldots, A_N$. The purpose of this article is to answer this question negatively.

In order to state our results we require an additional definition. Let us say that a pair of matrices (A_1, A_2) is $(c, \varepsilon, \lambda)$-resistant if it has the following property: for all choices of $i_1, \ldots, i_n \in \{1, 2\}$ such that at most εn of the integers i_k are equal to 2, we have $\|A_{i_1} \cdots A_{i_n}\| \geq c\lambda^n$. We will say that (A_1, A_2) *resists impurities*, or more simply is *resistant*, if it is $(c, \varepsilon, \lambda)$-resistant for some $c, \varepsilon > 0$ and some $\lambda > 1$. We recall the following conjecture of Bochi and Fayad [6]:

Conjecture 1.2. (Bochi–Fayad Conjecture) Let $\mathcal{H}$ denote the set of all 2×2 real matrices with unit determinant and unequal real eigenvalues, and let $\mathcal{E}$ denote the set of all 2×2 real matrices with unit determinant and non-real eigenvalues. Then the set of resistant pairs $(A_1, A_2) \in \mathcal{H} \times \mathcal{E}$ has full Lebesgue measure.

Some partial results in the direction of Conjecture 1.2 may be found in [1, 2, 14]. Bonatti has constructed explicit examples of resistant pairs in which A_2 is a rational rotation, and these examples are described in [7].

When investigating the discontinuities of $\mathfrak{r}_q$ we will focus on the situation in which $(A_1, \ldots, A_N)$ is a pair of real matrices of dimension two. In this case the probability vector $\mathsf{p} = (p_1, p_2)$ has the form $(p, 1-p)$ for some real number $p \in (0, 1)$, and in view of this we shall simply write $\mathfrak{r}_q(A_1, A_2, p)$ in place of the value $\mathfrak{r}_q(A_1, A_2, (p, 1-p))$ defined previously. We prove:

Theorem 1.3. *The function $\mathfrak{r}$ admits the following discontinuities:*

(i) *Let $q > 1$, $p \in (0, 1)$ and $0 < \delta < \lambda < \frac{1}{2}$, and suppose that δ is small enough that*
$$\frac{\log \min\{p^q, (1-p)^q\}}{\log \sqrt{\lambda\delta}} < \frac{\log(p^q + (1-p)^q)}{\log \lambda}.$$
Then the function $(A_1, A_2) \mapsto \mathfrak{r}_q(A_1, A_2, p)$ is discontinuous at the pair
$$A_1 := \begin{pmatrix} \lambda & 0 \\ 0 & \delta \end{pmatrix}, \quad A_2 := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

(ii) *If the Bochi–Fayad Conjecture is true then there exists a set $X \subseteq M_2(\mathbf{R})^2$ with positive Lebesgue measure with the following properties: $\|A_1\|, \|A_2\| < \frac{1}{2}$ for all $(A_1, A_2) \in X$, and there exists $Q > 1$ such that for all $p \in [\frac{1}{2}, 1)$ and*

$q > Q$, the function $(A_1, A_2) \mapsto \mathfrak{r}_q(A_1, A_2, p)$ is discontinuous at every point of X.

Remark. The reader will see from the proofs below that Theorem 1.3(i) in fact has the following more precise statement: if R_θ denotes the matrix corresponding to rotation about the origin through angle θ, then

$$\liminf_{\theta \to 0} \mathfrak{r}_q(A_1, \lambda R_\theta, p) < \mathfrak{r}_q(A_1, A_2, p).$$

Clearly the case of $p < \frac{1}{2}$ may also be considered in (ii) by interchanging the rôles of the matrices A_1 and A_2.

It would be of interest to remove the restriction on q in (ii) so as to bring that statement into line with (i). Unfortunately the Bochi–Fayad Conjecture does not seem to be a sufficiently strong statement to allow us to deduce that $(A_1, A_2) \mapsto \mathfrak{r}_q(A_1, A_2, p)$ has a positive-measure set of discontinuities for every $p \in (0, 1)$ and $q > 1$. We nonetheless conjecture that this map has a positive-measure set of discontinuities for all such p and q, and hope that whatever methods may be employed to prove the Bochi–Fayad Conjecture will also suffice to establish the discontinuity of $\mathfrak{r}_q(\cdot, \cdot, p)$ on a set of positive measure for all $q > 1$ and $p \in (0, 1)$.

2. Proof of Theorem 1.3

For $p \in (0, 1)$, $q > 1$, $s > 0$ and invertible matrices $A_1, A_2 \in M_2(\mathbf{R})$, let us define $p_1 := p$ and $p_2 := (1 - p)$, and write

$$\mathbf{R}_q(A_1, A_2, p, s) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1,\ldots,i_n=1}^{N} \varphi^s \left(A_{i_1} \cdots A_{i_n}\right)^{1-q} p_{i_1}^q \cdots p_{i_n}^q \right).$$

We note:

Lemma 2.1. *For fixed invertible matrices $A_1, A_2 \in M_d(\mathbf{R})$ such that $\|A_1\|$, $\|A_2\| < 1$, fixed $q > 1$ and fixed $p \in (0, 1)$ the function $\mathbf{R}_q(A_1, A_2, p, \cdot)\colon (0, +\infty) \to \mathbf{R}$ is well-defined and strictly increasing.*

Proof. It is well-known that $\varphi^s(AB) \le \varphi^s(A)\varphi^s(B)$ for all $s > 0$ and $A, B \in M_d(\mathbf{R})$, see for example [8, Lemma 2.1]; since the proof is brief we include it. For $s \ge d$ the result is trivial, and for $k \le s < k + 1$, $k = 0, \ldots, d - 1$ we have

$$\begin{aligned}
\varphi^s(AB) &= (\sigma_1(AB) \cdots \sigma_{k+1}(AB))^{s-k} (\sigma_1(AB) \cdots \sigma_k(AB))^{k+1-s} \\
&= \left\|\wedge^{k+1}(AB)\right\|^{s-k} \left\|\wedge^k(AB)\right\|^{k+1-s} \\
&\le \left\|\wedge^{k+1}A\right\|^{s-k} \left\|\wedge^{k+1}B\right\|^{s-k} \left\|\wedge^k A\right\|^{k+1-s} \left\|\wedge^k B\right\|^{k+1-s} = \varphi^s(A)\varphi^s(B)
\end{aligned}$$

as claimed (where $\wedge^k A$ denotes the k^{th} exterior power of A). It follows that each sequence (a_n) defined by

$$a_n := \log \left(\sum_{i_1,\ldots,i_n=1}^{N} \varphi^s \left(A_{i_1} \cdots A_{i_n}\right)^{1-q} p_{i_1}^q \cdots p_{i_n}^q \right)$$

satisfies $a_{n+m} \ge a_n + a_m$ for all $n, m \ge 1$, and this is well known to imply the convergence of the sequence $(1/n)a_n$ to a limit in $(-\infty, +\infty]$. Observe that $\varphi^s(A) \ge \sigma_2(A)^s$ for all $A \in M_2(\mathbf{R})$. Since A_1, A_2 are invertible we have $\sigma_2(A_1), \sigma_2(A_2) \ge \varepsilon$ for some $\varepsilon > 0$, and thus

$$\begin{aligned}\sum_{i_1,\ldots,i_n=1}^{N} \varphi^s\left(A_{i_1}\cdots A_{i_n}\right)^{1-q} p_{i_1}^q \cdots p_{i_n}^q &\leq \sum_{i_1,\ldots,i_n=1}^{N} \sigma_2\left(A_{i_1}\cdots A_{i_n}\right)^{s(1-q)} p_{i_1}^q \cdots p_{i_n}^q \\ &\leq \sum_{i_1,\ldots,i_n=1}^{N} \varepsilon^{s(1-q)} p_{i_1}^q \cdots p_{i_n}^q \\ &= \varepsilon^{ns(1-q)}\left(p^q + (1-p)^q\right)^n\end{aligned}$$

(where we have used the fact that $1-q$ is negative) so that the limit is finite.

Let us show that $\mathbf{R}_q(A_1, A_2, p, s)$ is strictly increasing in s. We note that $\varphi^{s+t}(A) \leq \varphi^s(A)\|A\|^t$ for all $s, t > 0$ and for every matrix $A \in M_2(\mathbf{R})$. Taking $\theta := \max\{\|A_1\|, \|A_2\|\} \in (0,1)$ it follows that for all $n \geq 1$

$$\sum_{i_1,\ldots,i_n=1}^{N} \varphi^{s+t}\left(A_{i_1}\cdots A_{i_n}\right)^{1-q} p_{i_1}^q \cdots p_{i_n}^q \geq \theta^{nt(1-q)} \sum_{i_1,\ldots,i_n=1}^{N} \varphi^s\left(A_{i_1}\cdots A_{i_n}\right)^{1-q} p_{i_1}^q \cdots p_{i_n}^q$$

and therefore

$$\mathbf{R}_q(A_1, A_2, p, s+t) \geq (1-q)t\log\theta + \mathbf{R}_q(A_1, A_2, p, s) > \mathbf{R}_q(A_1, A_2, p, s)$$

as required. □

Our interest in the previous lemma is due to the following consequence:

Corollary 2.2. *For all invertible matrices A_1, A_2 such that $\|A_1\|, \|A_2\| < 1$, all $p \in (0,1)$ and all $q > 1$, we have*

$$\begin{aligned}\mathfrak{r}_q(A_1, A_2, p) &= \sup\left\{s > 0\colon \mathbf{R}_q(A_1, A_2, p, s) < 0\right\} \\ &= \inf\left\{s > 0\colon \mathbf{R}_q(A_1, A_2, p, s) > 0\right\}.\end{aligned} \tag{1}$$

Before commencing the proof of Theorem 1.3, let us briefly describe its strategy. The proofs of continuity of the affinity dimension $\mathfrak{s}$ given in [15, 24] operate by defining the singular value pressure function

$$\mathbf{S}(A_1, \ldots, A_N, s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1,\ldots,i_n=1}^{N} \varphi^s\left(A_{i_1}\cdots A_{i_n}\right)$$

and observing that for fixed invertible $A_1, \ldots, A_N$ with $\max\|A_i\| < 1$ the function $s \mapsto \mathbf{S}(A_1, \ldots, A_N, s)$ is strictly decreasing, so that

$$\begin{aligned}\mathfrak{s}(A_1, \ldots, A_N) &= \sup\{s > 0\colon \mathbf{S}(A_1, \ldots, A_N, s) > 0\} \\ &= \inf\{s > 0\colon \mathbf{S}(A_1, \ldots, A_N, s) < 0\}.\end{aligned} \tag{2}$$

The proofs then proceed by showing that for each fixed $s > 0$ (or in the case of [24], for a dense set of $s > 0$) the function $(A_1, \ldots, A_N) \mapsto \mathbf{S}(A_1, \ldots, A_N, s)$ is continuous, and then deduce the continuity of $\mathfrak{s}$ via the formula (2). The argument which we employ in proving Theorem 1.3 essentially converse to this: we demonstrate the existence of discontinuities in the function $(A_1, A_2) \mapsto \mathbf{R}_q(A_1, A_2, p, s)$ and show that they induce discontinuities in $\mathfrak{r}_q$ via the equation (1).

The origin of these discontinuities can be described informally as follows. Following [17], let us define the *lower spectral radius* of a pair of matrices A_1, A_2 to be the quantity

$$\underline{\varrho}(A_1, A_2) := \lim_{n\to\infty} \min_{1\leq i_1,\ldots,i_n\leq 2} \|A_{i_1}\cdots A_{i_n}\|^{\frac{1}{n}} = \inf_{n\geq 1} \min_{1\leq i_1,\ldots,i_n\leq 2} \|A_{i_1}\cdots A_{i_n}\|^{\frac{1}{n}}.$$

The lower spectral radius is known to depend discontinuously on the matrix entries in general [20, p. 20], and this phenomenon was investigated in depth by the author and Bochi in [7]. This relates to $\mathbf{R}_q(A_1, A_2, p, s)$ as follows: if $0 < s \leq 1$ then we may estimate

$$\begin{aligned}\mathbf{R}_q(A_1, A_2, p, s) &\leq \limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1,\ldots,i_n=1}^{2} p_{i_1}^q \cdots p_{i_n}^q \max_{1\leq j_1,\ldots,j_n\leq 2} \|A_{j_1}\cdots A_{j_n}\|^{s(1-q)} \right)\\ &= \log(p^q + (1-p)^q) + s(1-q)\log \underline{\varrho}(A_1, A_2)\end{aligned}$$

—where the negativity of the exponent $s(1-q)$ has the critical effect of converting the maximum over all matrix products into a minimum—and similarly on the other hand

$$\begin{aligned}\mathbf{R}_q(A_1, A_2, p, s) &\geq \liminf_{n\to\infty} \frac{1}{n} \log \left(\min_{1\leq i_1,\ldots,i_n\leq 2} p_{i_1}^q \cdots p_{i_n}^q \cdot \max_{1\leq j_1,\ldots,j_n\leq 2} \|A_{j_1}\cdots A_{j_n}\|^{s(1-q)} \right)\\ &= q\log\min\{p, 1-p\} + s(1-q)\log \underline{\varrho}(A_1, A_2).\end{aligned}$$

These estimates, despite their crudity, imply that if the discontinuity of the lower spectral radius $\underline{\varrho}$ at a particular pair of matrices is strong enough then it induces a discontinuity in the function $\mathbf{R}_q$, which can be exploited to deduce a discontinuity in the function $\mathfrak{r}_q$. Indeed, the examples of discontinuity of $\mathfrak{r}_q$ in Theorem 1.3(i) and (ii) correspond directly with known examples of the discontinuity of the lower spectral radius, specifically Example 1.1 and Proposition 7.6 in [7]. In the context of Theorem 1.3(i) we can obtain sufficient control on the size of the discontinuity in $\underline{\varrho}$ without any assumptions on p and q. In the context of Theorem 1.3(ii) our much weaker control on the discontinuities of $\underline{\varrho}$ means that the above estimate is only useful if q is large and p is close to $\frac{1}{2}$, which has the effect of bringing the quantities $\log(p^q + (1-p)^q)$ and $q\log(\min\{p, (1-p)\})$ closer together. In order to deal with more general p the proof of Theorem 1.3(ii) in fact applies a slightly finer estimate than that indicated above: for this we require a slightly strengthened statement of [7, Proposition 7.6], which shows not only *that* the lower spectral radius is discontinuous in certain places but also specifies *how* it is discontinuous. We nonetheless emphasise that the conceptual origin of the discontinuity of $\mathfrak{r}_q$ in this paper is that it is a consequence of the discontinuity of the lower spectral radius.

Proof of Theorem 1.3(i). Let $q > 1$ and $0 < \lambda < \frac{1}{2}$, and let A_1, A_2, δ be as in Theorem 1.3(i). Throughout the proof we shall find it useful to write $p_1 := p$, $p_2 := (1-p)$ in order to simplify certain frequently-arising expressions. We observe that by straightforward differentiation and minimisation with respect to p one has $p^q + (1-p)^q \geq 2^{1-q}$ for every $q > 1$ and $p \in (0, 1)$. In particular, noting the hypothesis of Theorem 1.3(i) and the negativity of $\log \lambda$,

$$0 < \frac{q\log\min\{p, 1-p\}}{(q-1)\log\sqrt{\lambda\delta}} < \frac{\log(p^q + (1-p)^q)}{(q-1)\log\lambda} \leq \frac{\log\frac{1}{2}}{\log\lambda} < 1. \tag{3}$$

Clearly $\|A_{i_1}\cdots A_{i_n}\| = \lambda^n$ for every $i_1, \ldots, i_n \in \{1, 2\}$ and $n \geq 1$, so for every $s \in (0, 1]$

$$\begin{aligned}\mathbf{R}_q(A_1, A_2, p, s) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1,\ldots,i_n=1}^{2} \lambda^{sn(1-q)} p_{i_1}^q \cdots p_{i_n}^q \right)\\ &= s(1-q)\log\lambda + \log\left(p^q + (1-p)^q\right).\end{aligned}$$

In particular if $s < \log(p^q + (1-p)^q)/(q-1)\log\lambda \in (0,1]$ then $\mathbf{R}_q(B_1, B_2, p, s) < 0$, so we have

$$\mathfrak{r}_q(A_1, A_2, p) \geq \frac{\log(p^q + (1-p)^q)}{(q-1)\log\lambda}$$

using Corollary 2.2. Now fix an integer $k \geq 1$ and define $B_1 := A_1$ and

$$B_2 := \lambda \begin{pmatrix} \cos\frac{\pi}{2k} & -\sin\frac{\pi}{2k} \\ \sin\frac{\pi}{2k} & \cos\frac{\pi}{2k} \end{pmatrix}$$

so that

$$B_2^k = \begin{pmatrix} 0 & -\lambda^k \\ \lambda^k & 0 \end{pmatrix}.$$

(We observe that by taking k sufficiently large, (B_1, B_2) may be taken as close to (A_1, A_2) as desired.) Since we have

$$B_1^n B_2^k B_1^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \delta^n \end{pmatrix} \begin{pmatrix} 0 & -\lambda^k \\ \lambda^k & 0 \end{pmatrix} \begin{pmatrix} \lambda^n & 0 \\ 0 & \delta^n \end{pmatrix} = \begin{pmatrix} \lambda^{n+k}\delta^n & 0 \\ 0 & \lambda^{n+k}\delta^n \end{pmatrix}$$

for all $n \geq 1$ it follows that

$$\min_{1 \leq i_1, \ldots, i_{2n+k} \leq 2} \|B_{i_1} \cdots B_{i_n}\| \leq \lambda^{n+k}\delta^n$$

for every $n \geq 1$, and therefore

$$\lim_{n\to\infty} \left(\min_{1 \leq i_1, \ldots, i_n \leq 2} \|B_{i_1} \cdots B_{i_n}\| \right)^{\frac{1}{n}} \leq \sqrt{\lambda\delta}.$$

Hence for $0 < s \leq 1$

$$\begin{aligned}
\mathbf{R}_q(B_1, B_2, p, s) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1, \ldots, i_n = 1}^{2} \|B_{i_1} \cdots B_{i_n}\|^{s(1-q)} p_{i_1}^q \cdots p_{i_n}^q \right) \\
&\geq \lim_{n\to\infty} \frac{1}{n} \log \left(\max_{1 \leq i_1, \ldots, i_n \leq 2} \left(\|B_{i_1} \cdots B_{i_n}\|^{s(1-q)} \right) \min\{p^{qn}, (1-p)^{qn}\} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(\left(\min_{1 \leq i_1, \ldots, i_n \leq 2} \|B_{i_1} \cdots B_{i_n}\| \right)^{s(1-q)} \min\{p^{qn}, (1-p)^{qn}\} \right) \\
&\geq \frac{s(1-q)}{2} \log(\lambda\delta) + \log\min\{p^q, (1-p)^q\}.
\end{aligned}$$

If $1 \geq s > q\log\min\{p, 1-p\}/(q-1)\log\sqrt{\lambda\delta} \in (0,1)$ then this last term exceeds 0 and therefore $\mathfrak{r}_q(B_1, B_2, p) \geq s$. Hence in view of Corollary 2.2

$$\mathfrak{r}_q(B_1, B_2, p) \leq \frac{q\log\min\{p, 1-p\}}{(q-1)\log\sqrt{\lambda\delta}}.$$

As was previously noted, by taking the integer k in the definition of B_2 arbitrarily large we may take (B_1, B_2) as above arbitrarily close to (A_1, A_2), and it follows that

$$\begin{aligned}
\liminf_{(B_1,B_2)\to(A_1,A_2)} \mathfrak{r}_q(B_1, B_2, p) &\leq \frac{2q\log\min\{p, 1-p\}}{(q-1)\log(\lambda\delta)} \\
&< \frac{\log(p^q + (1-p)^q)}{(q-1)\log\lambda} \leq \mathfrak{r}_q(A_1, A_2, p)
\end{aligned}$$

where we have used (3), so that $\mathfrak{r}_q$ is discontinuous at (A_1, A_2) as claimed. This completes the proof of (i). □

The proof of (ii) uses closely analogous estimates, but we require an additional result relating to Conjecture 1.2. The following result is a more specialised reworking of one half of [7, Proposition 7.6].

Lemma 2.3. *Suppose that Conjecture 1.2 is true. Then there exist $\varepsilon,\kappa>0$ and a set $X\subset M_2(\mathbf{R})^2$ with positive Lebesgue measure such that for all $(A_1,A_2)\in X$ we have*

$$\|A_1\|,\|A_2\|<\frac{1}{2},\quad \underline{\varrho}(A_1,A_2)\geq e^{-\kappa},$$

but such that in every open neighbourhood of (A_1,A_2) we may find (B_1,B_2) such that for a certain integer $k\geq 1$ depending on (B_1,B_2)

$$\lim_{n\to\infty}\left\|B_1^nB_2^kB_1^n\right\|^{\frac{1}{2n+k}}=\underline{\varrho}(B_1,B_2)\leq e^{-\varepsilon-\kappa}.$$

Proof. Let us first define

$$Z:=\left\{(\alpha H,\beta R)\colon H\in\mathcal{H},R\in\mathcal{E}\text{ and }0<\alpha<\beta<\frac{1}{2\|H\|}.\right\}$$

Clearly this is an open subset of $M_2(\mathbf{R})^2$, and for every $(A_1,A_2)\in Z$ both of the matrices A_i have positive determinant and have norm strictly less than one half. By the hypothesis that Conjecture 1.2 is true, the set of all $(\alpha H,\beta R)\in Z$ such that (H,R) is resistant has full Lebesgue measure in Z, and hence in particular has positive Lebesgue measure in $M_2(\mathbf{R})^2$.

We first claim that for all $(A_1,A_2)=(\alpha H,\beta R)$ such that (H,R) is resistant we have $\underline{\varrho}(A_1,A_2)>\sqrt{\det A_1}$. Indeed, suppose that (H,R) is (c,λ,ε)-resistant where $c,\varepsilon>0$ and $\lambda>1$. If $A_{i_1},\ldots,A_{i_n}$ contains at most εn instances of A_2 then we have

$$\|A_{i_1}\cdots A_{i_n}\|\geq c\lambda^n\alpha^n$$

since $\|A_{i_1}\cdots A_{i_n}\|$ is at least α^n times the norm of a product of n of the matrices H,R in which at most εn matrices are equal to R. On the other hand if the product $A_{i_1}\cdots A_{i_n}$ contains at least εn instances of A_2 then we have

$$\|A_{i_1}\cdots A_{i_n}\|\geq\sqrt{|\det A_{i_1}\cdots A_{i_n}|}\geq\alpha^{(1-\varepsilon)n}\beta^{\varepsilon n},$$

and therefore

$$\underline{\varrho}(A_1,A_2)=\lim_{n\to\infty}\inf_{1\leq i_1,\ldots,i_n\leq 2}\|A_{i_1}\cdots A_{i_n}\|^{\frac{1}{n}}\geq\min\left\{\lambda\alpha,\left(\frac{\beta}{\alpha}\right)^{\varepsilon}\alpha\right\}>\alpha=\sqrt{\det A_1}$$

as claimed.

We next claim that for every $(A_1,A_2)=(\alpha H,\beta R)\in Z$, we may in every open neighbourhood of A_2 find a matrix B_2 such that for some integer $k\geq1$ depending on B_2,

$$\lim_{n\to\infty}\left\|A_1^nB_2^kA_1^n\right\|^{\frac{1}{2n+k}}=\sqrt{\det A_1}.$$

(We note that in this case necessarily $\underline{\varrho}(A_1,B_2)=\sqrt{\det A_1}$, since clearly any product of n of those two matrices is bounded below in norm by the square root of the determinant, which in turn is bounded below by $(\det A_1)^{n/2}$.) To show this it is sufficient to show that for any fixed $H\in\mathcal{H}$ and $R\in\mathcal{E}$, we may in every open neighbourhood of R find a matrix R' such that for some integer $k\geq1$ depending on R',

$$\lim_{n\to\infty}\left\|H^n(R')^kH^n\right\|^{\frac{1}{2n+k}}=1. \tag{4}$$

Let us prove this statement. Given $(H,R) \in \mathcal{H} \times \mathcal{E}$ let $\lambda > 1$ denote the larger eigenvalue of H, and let u and v be eigenvectors of H corresponding respectively to the eigenvalues λ and λ^{-1}. Since R has non-real eigenvalues it is conjugate to a rotation through some particular angle θ. It is easy to see that this implies that in any neighbourhood of R we may find a matrix R', conjugate to a rotation through a different angle, such that $(R')^k u = \gamma v$ for some nonzero real number γ and integer $k \geq 1$. In the basis (u, v) we have

$$H^n (R')^k H^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix} \begin{pmatrix} 0 & \delta \\ \gamma & \epsilon \end{pmatrix} \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix} = \begin{pmatrix} 0 & \delta \\ \gamma & \lambda^{-2n}\epsilon \end{pmatrix}$$

for some real numbers δ, ϵ, where the first column of $(R')^k$ reflects the fact that $(R')^k u = \gamma v$. In particular $\|H^n(R')^k H^n\|$ is bounded independently of n, and this yields (4).

Summarising the proof so far, we have shown that there is a full-measure subset Z_1 of the open set $Z \subset M_2(\mathbf{R})^2$ such that every $(A_1, A_2) \in Z_1$ satisfies $\|A_1\|, \|A_2\| < \frac{1}{2}$ and $\underline{\varrho}(A_1, A_2) > \sqrt{\det A_1}$, and has the property that in every open neighbourhood of A_2 we may find B_2 such that for some integer $k \geq 1$,

$$\lim_{n\to\infty} \left\|A_1^n B_2^k A_1^n\right\|^{\frac{1}{2n+k}} = \underline{\varrho}(A_1, B_2) = \sqrt{\det A_1}.$$

So, let us choose $\kappa > 0$ such that the set

$$Z_2 := \left\{(A_1, A_2) \in Z_1 \colon \underline{\varrho}(A_1, A_1) \geq e^{-\kappa} > \sqrt{\det A_1}\right\}$$

has positive Lebesgue measure, and choose $\varepsilon > 0$ such that the set

$$X := \left\{(A_1, A_2) \in Z_2 \colon \sqrt{\det A_1} < e^{-\kappa-\varepsilon}\right\}$$

has positive Lebesgue measure. The proof is complete. □

Remark. In order to improve Theorem 1.3(ii) so as to allow arbitrary $q > 1$ it would be sufficient to be able to choose the set X in such a way that the ratio ε/κ is made arbitrarily large. In effect, this asks that we should be able to reduce the second singular value of A_1 arbitrarily far without simultaneously reducing $\underline{\varrho}(A_1, A_2)$ by a comparable amount: in Theorem 1.3(i), this effect is achieved by the simple expedient of reducing δ.

Proof of Theorem 1.3(ii). Let X, ε, κ be as in Lemma 2.3 and choose $Q := 1 + \frac{\kappa}{\varepsilon} > 1$. For all $p \in [\frac{1}{2}, 1)$ and $q > Q$ we have

$$(p^q + (1-p)^q)^{\frac{1}{q}} < p^{\frac{\kappa}{\kappa+\varepsilon}}$$

since for each fixed $q > 1$ the former expression is a convex function of p, the latter is a concave function of p, the two functions agree at $p = 1$ and the former function is strictly less than the latter at $p = \frac{1}{2}$. Rearranging we find that for all such p and q

$$\frac{\log p^q}{\log(p^q + (1-p)^q)} < 1 + \frac{\varepsilon}{\kappa}.$$

Since clearly $e^{-\kappa} \leq \underline{\varrho}(A_1, A_2) \leq \max\{\|A_1\|, \|A_2\|\} < \frac{1}{2}$ we have $\kappa > \log 2$ and therefore

$$(5) \qquad 0 < \frac{\log(p^q)}{(1-q)(\varepsilon+\kappa)} < \frac{\log(p^q + (1-p)^q)}{(1-q)\kappa} < \frac{\log(p^q + (1-p)^q)}{(1-q)\log 2} \leq 1$$

for all $p \in [\frac{1}{2}, 1)$ and $q > Q$, where we have reused the elementary inequality $p^q + (1-p)^q \geq 2^{1-q}$ which was similarly applied in (i). We will show that for all such p and q, every point of X is a point of discontinuity of the map $(A_1, A_2) \mapsto \mathfrak{r}_q(A_1, A_2, p)$.

Let us therefore fix p and q and take $(A_1, A_2) \in X$. For every $s \in (0, 1]$ we have

$$\begin{aligned}\mathbf{R}_q(A_1, A_2, p, s) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1,\ldots,i_n=1}^{2} \|A_{i_1} \cdots A_{i_n}\|^{s(1-q)} p_{i_1}^q \cdots p_{i_n}^q \right) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\max_{1\leq i_1,\ldots,i_n\leq 2} \left(\|A_{i_1} \cdots A_{i_n}\|^{s(1-q)}\right) \sum_{j_1,\ldots,j_n=1}^{2} p_{j_1}^q \cdots p_{j_n}^q \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\left(\min_{1\leq i_1,\ldots,i_n\leq 2} \|A_{i_1} \cdots A_{i_n}\| \right)^{s(1-q)} \sum_{j_1,\ldots,j_n=1}^{2} p_{j_1}^q \cdots p_{j_n}^q \right) \\ &= s(1-q) \log \underline{\varrho}(A_1, A_2) + \log\left(p^q + (1-p)^q\right) \\ &\leq s(q-1)\kappa + \log\left(p^q + (1-p)^q\right).\end{aligned}$$

It follows that if $s < \log(p^q + (1-p)^q)/(1-q)\kappa < 1$ then $\mathbf{R}_q(A_1, A_2, p, s)$ is negative, and hence by Corollary 2.2

$$\mathfrak{r}_q(A_1, A_2, p) \geq \frac{\log(p^q + (1-p)^q)}{(1-q)\kappa}.$$

On the other hand we may take (B_1, B_2) arbitrarily close to (A_1, A_2) such that

$$\lim_{n\to\infty} \left\|B_1^n B_2^k B_1^n\right\|^{\frac{1}{2n+k}} = \underline{\varrho}(B_1, B_2) < e^{-\varepsilon-\kappa}$$

for some integer $k \geq 1$. In particular we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} \log \max_{1\leq i_1,\ldots,i_n\leq 2} \left(\|B_{i_1} \cdots B_{i_n}\|^{s(1-q)} p_{i_1}^q \cdots p_{i_n}^q \right) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log \max_{1\leq i_1,\ldots,i_n\leq 2} \left(\|B_{i_1} \cdots B_{i_n}\|\right)^{s(1-q)} + \lim_{n\to\infty} \frac{1}{n} \log \max_{1\leq i_1,\ldots,i_n\leq 2} \left(p_{i_1}^q \cdots p_{i_n}^q\right) \\ &= \lim_{n\to\infty} \frac{s(1-q)}{n} \log \min_{1\leq i_1,\ldots,i_n\leq 2} \|B_{i_1} \cdots B_{i_n}\| + q \log p \\ &= s(1-q) \log \underline{\varrho}(B_1, B_2) + q \log p\end{aligned}$$

since $p_1 := p \geq \frac{1}{2} \geq 1 - p = p_2$, but also

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} \log \max_{1\leq i_1,\ldots,i_n\leq 2} \left(\|B_{i_1} \cdots B_{i_n}\|^{s(1-q)} p_{i_1}^q \cdots p_{i_n}^q \right) \\ &\geq \lim_{n\to\infty} \frac{1}{2n+k} \log \left(\left\|B_1^n B_2^k B_1^n\right\|^{s(1-q)} p_1^n p_2^k p_1^n \right) \\ &= s(q-1) \log \underline{\varrho}(B_1, B_2) + q \log p,\end{aligned}$$

and we conclude that the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \max_{1\leq i_1,\ldots,i_n\leq 2} \left(\|B_{i_1} \cdots B_{i_n}\|^{s(1-q)} p_{i_1}^q \cdots p_{i_n}^q \right)$$

is equal to $s(q-1)\log \underline{\varrho}(B_1,B_2)+q\log p$. It follows that for all $s\in(0,1]$

$$\begin{aligned}\mathbf{R}_q(B_1,B_2,p,s) &= \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i_1,\dots,i_n=1}^{2}\|B_{i_1}\cdots B_{i_n}\|^{s(1-q)}p_{i_1}^q\cdots p_{i_n}^q\right)\\ &\geq \lim_{n\to\infty}\frac{1}{n}\log\left(\max_{1\leq i_1,\dots,i_n\leq 2}\left(\|B_{i_1}\cdots B_{i_n}\|^{s(1-q)}p_{i_1}^q\cdots p_{i_n}^q\right)\right)\\ &= s(1-q)\log\underline{\varrho}(B_1,B_2)+q\log p\\ &> s(q-1)(\varepsilon+\kappa)+q\log p.\end{aligned}$$

If $1\geq s>(q\log p)/(1-q)(\varepsilon+\kappa)\in(0,1)$ then $\mathbf{R}_q(B_1,B_2,p,s)>0$, and therefore

$$\mathfrak{r}_q(B_1,B_2,p)\leq\frac{\log(p^q)}{(1-q)(\varepsilon+\kappa)}$$

by Corollary 2.2. Hence

$$\begin{aligned}\liminf_{(B_1,B_2)\to(A_1,A_2)}\mathfrak{r}_q(B_1,B_2,p) &\leq \frac{q\log p}{(1-q)(\varepsilon+\kappa)}\\ &< \frac{\log(p^q+(1-p)^q)}{(1-q)\kappa}\\ &\leq \mathfrak{r}_q(A_1,A_2,p)\end{aligned}$$

using (5), and $(B_1,B_2)\mapsto\mathfrak{r}_q(B_1,B_2,p)$ is discontinuous at (A_1,A_2) as claimed. □

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