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MAPPINGS PRESERVING SEGAL'S ENTROPY IN VON NEUMANN ALGEBRAS

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Abstract. We investigate the situation when a normal positive linear unital map on a semifinite von Neumann algebra leaving the trace invariant does not change the Segal entropy of the density of a normal, not necessarily normalised, state. Two cases are dealt with: a) no restriction on the map is imposed, b) the map represents a repeatable instrument in measurement theory which means that it is idempotent.

Introduction

In the paper, the question of invariance of Segal's entropy under the action of a normal positive linear unital map is addressed in the case of a semifinite von Neumann algebra.

The notion of Segal's entropy was introduced by Segal in [9] for semifinite von Neumann algebras as a direct counterpart of von Neumann's entropy defined for the full algebra $\mathbf{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space by means of the canonical trace. However, in the case of an arbitrary semifinite von Neumann algebra, where instead of the canonical trace we have a normal semifinite faithful trace, substantial differences between these two entropies arise. Perhaps the most fundamental one consists in the fact that while a normal state on $\mathbf{B}(\mathcal{H})$ is represented by a positive operator of trace one (the so-called 'density matrix'), in the case of an arbitrary semifinite von Neumann algebra this 'density matrix' can be an unbounded operator. This prompted Segal to consider only the states whose 'density matrices' were in the algebra. In our analysis, we avoid this restriction as well as we allow the trace to be semifinite and not finite, the latter being also often assumed while dealing with Segal's entropy.

On the way to the main theorems, some auxiliary results about strict operator convexity or Jensen's inequality for unbounded measurable operators are obtained which seem to be interesting and of some importance in their own right.

1. Preliminaries and notation

Let $\mathscr{M}$ be a semifinite von Neumann algebra of operators acting on a Hilbert space $\mathcal{H}$ with a normal semifinite faithful trace τ, identity $\mathbf{1}$, and predual $\mathscr{M}_*$. By $\mathscr{M}^+$ we shall denote the set of positive operators in $\mathscr{M}$, and by $\mathscr{M}_*^+$—the set of positive functionals in $\mathscr{M}_*$. These functionals will be sometimes referred to as (non-normalised) states.

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The algebra of *measurable operators* $\widetilde{\mathscr{M}}$ is defined as a topological *-algebra of densely defined closed operators on $\mathcal{H}$ affiliated with $\mathscr{M}$ with strong addition $\dot{+}$ and strong multiplication $\cdot$, i.e.

$$x \dot{+} y = \overline{x+y}, \quad x \cdot y = \overline{xy}, \quad x, y \in \widetilde{\mathscr{M}},$$

where $\overline{x+y}$ and $\overline{xy}$ are the closures of the corresponding operators defined by addition and composition respectively on the natural domains given by the intersections of the domains of the x and y and of the range of y and the domain of x respectively. In what follows, we shall omit the dot in the symbols of these operations and write simply $x+y$ and xy to denote $x \dot{+} y$ and $x \cdot y$.

The domain of a linear operator x on $\mathcal{H}$ will be denoted by $\mathcal{D}(x)$.

For each $\rho \in \mathscr{M}_*$, there is a measurable operator h such that

$$\rho(x) = \tau(xh) = \tau(hx), \quad x \in \mathscr{M}.$$

The space of all such operators is denoted by $L^1(\mathscr{M}, \tau)$, and the correspondence above is one-to-one and isometric, where the norm on $L^1(\mathscr{M}, \tau)$, denoted by $\|\cdot\|_1$, is defined as

$$\|h\|_1 = \tau(|h|), \quad h \in L^1(\mathscr{M}, \tau).$$

(In the theory of noncommutative L^p-spaces for semifinite von Neumann algebras, it it shown that τ can be extended to the h's as above; see e.g. [5, 10, 11] for a detailed account of this theory.) Moreover, to hermitian functionals in $\mathscr{M}_*$ correspond selfadjoint operators in $L^1(\mathscr{M}, \tau)$, and to states in $\mathscr{M}_*$—positive operators in $L^1(\mathscr{M}, \tau)$.

For $\rho \in \mathscr{M}_*^+$, the corresponding element in $L^1(\mathscr{M}, \tau)$ is called the *density* of ρ and is denoted by h_ρ. The *Segal entropy* of ρ, denoted by $H(\rho)$, is defined as

$$H(\rho) = \tau(h_\rho \log h_\rho),$$

i.e. for the spectral representation of h_ρ

$$h_\rho = \int_0^\infty t\, e(dt), \tag{1}$$

we have

$$H(\rho) = \int_0^\infty t \log t\, \tau(e(dt)).$$

Accordingly, we define Segal's entropy for $0 \leqslant h \in L^1(\mathscr{M}, \tau)$ by the formula

$$H(h) = \tau(h \log h) = \int_0^\infty t \log t\, \tau(e(dt)),$$

where h has the spectral representation as in (1). Let us note that the existence of Segal's entropy is by no means guaranteed, however, for finite τ and normal state ρ, we have, on account of the inequality

$$t \log t \geqslant t - 1,$$

the relation

$$\begin{aligned} H(\rho) &= \int_0^\infty t \log t\, \tau(e(dt)) \geqslant \int_0^\infty (t-1)\, \tau(e(dt)) \\ &= \tau\Big(\int_0^\infty t\, e(dt)\Big) - \tau\Big(\int_0^\infty e(dt)\Big) = \tau(h_\rho) - \tau(\mathbf{1}) > -\infty, \end{aligned}$$

showing that, at least in this case, Segal's entropy is well defined (and nonnegative for normalised states).

2. Operator convexity and Jensen's inequality

It is well-known that the function

$$g(t) = \frac{1}{s+t}$$

is strictly operator convex on $\mathbf{B}(\mathcal{H})^+$ for each positive s. However, we shall need more.

Proposition 1. *Let $\mathscr{M}$ be a semifinite von Neumann algebra. The function g as above is strictly operator convex on $\widetilde{\mathscr{M}}^+$, which means that for arbitrary $x_1, x_2 \in \widetilde{\mathscr{M}}^+$ and arbitrary $\lambda \in (0,1)$ we have*

$$\big(s\mathbf{1} + \lambda x_1 + (1-\lambda)x_2\big)^{-1} \leqslant \lambda\big(s\mathbf{1} + x_1\big)^{-1} + (1-\lambda)\big(s\mathbf{1} + x_2\big)^{-1} \tag{2}$$

with equality if and only if $x_1 = x_2$.

Proof. Let u be a positive selfadjoint measurable operator such that u^{-1} is also measurable. Then

$$0 \leqslant \big(u^{1/2} - u^{-1/2}\big)^2 = u + u^{-1} - 2\mathbf{1}.$$

For arbitrary $\lambda \in (0,1)$, we have

$$\lambda(1-\lambda)\big(u + u^{-1} - 2\mathbf{1}\big) \geqslant 0,$$

which yields the inequality

$$(1 - 2\lambda + 2\lambda^2)\mathbf{1} + \lambda(1-\lambda)u + \lambda(1-\lambda)u^{-1} \geqslant \mathbf{1},$$

i.e.

$$\big(\lambda\mathbf{1} + (1-\lambda)u\big)\big(\lambda\mathbf{1} + (1-\lambda)u^{-1}\big) \geqslant \mathbf{1}.$$

Multiplying both sides of the inequality above on the right and on the left by $\big(\lambda\mathbf{1} + (1-\lambda)u\big)^{-1/2}$, we get

$$\lambda\mathbf{1} + (1-\lambda)u^{-1} \geqslant \big(\lambda\mathbf{1} + (1-\lambda)u\big)^{-1}. \tag{3}$$

Assume now that u is of the form

$$u = z_1^{-1/2} z_2 z_1^{-1/2}, \tag{4}$$

where $z_1^{1/2}$, $z_1^{-1/2}$, z_2 and z_2^{-1} are positive measurable operators. Taking into account the equality

$$\big(\lambda\mathbf{1} + (1-\lambda)z_1^{-1/2} z_2 z_1^{-1/2}\big)^{-1} = \big(z_1^{-1/2}(\lambda z_1 + (1-\lambda)z_2)z_1^{-1/2}\big)^{-1},$$

we obtain from the inequality (3) the relation

$$\begin{aligned}\lambda\mathbf{1} + (1-\lambda)z_1^{1/2} z_2^{-1} z_1^{1/2} &\geqslant \big(\lambda\mathbf{1} + (1-\lambda)z_1^{-1/2} z_2 z_1^{-1/2}\big)^{-1} \\ &= \big(z_1^{-1/2}(\lambda z_1 + (1-\lambda)z_2)z_1^{-1/2}\big)^{-1} = z_1^{1/2}\big(\lambda z_1 + (1-\lambda)z_2\big)^{-1} z_1^{1/2}.\end{aligned}$$

Multiplying both sides of the inequality above on the right and on the left by $z_1^{-1/2}$, we obtain

$$\lambda z_1^{-1} + (1-\lambda)z_2^{-1} \geqslant \big(\lambda z_1 + (1-\lambda)z_2\big)^{-1}. \tag{5}$$

Now putting

$$z_1 = s\mathbf{1} + x_1, \quad z_2 = s\mathbf{1} + x_2,$$

we get $z_1^{1/2}, z_2 \in \widetilde{\mathscr{M}}$, $z_1^{-1/2}, z_2^{-1} \in \mathscr{M}$, thus u defined by the equality (4), as well as u^{-1}, are measurable, and the inequality (5) shows that the relation (2) holds. Now it is clear that equality in (2) holds if and only if $u^{1/2} = u^{-1/2}$, i.e. $u = \mathbf{1}$, which yields

$$z_1^{-1/2} z_2 z_1^{-1/2} = \mathbf{1},$$

meaning that $z_1 = z_2$, i.e. $x_1 = x_2$. □

The function

$$f(t) = t \log t, \quad t \in [0, \infty) \tag{6}$$

is known to be strictly operator convex for positive bounded operators. We want to extend this result to some class of unbounded operators as well. Observe that this property amounts to the inequality

$$\lambda x_1 \log x_1 + (1-\lambda) x_2 \log x_2 \geqslant \big(\lambda x_1 + (1-\lambda) x_2\big) \log \big(\lambda x_1 + (1-\lambda) x_2\big)$$

for positive selfadjoint operators x_1 and x_2, and arbitrary $\lambda \in (0,1)$ with equality if and only if $x_1 = x_2$. However, in such a general setting obvious problems arise with the domains or selfadjointness of respective operators. We shall confine ourselves to selfadjoint measurable operators where these problems can be avoided. Observe first that measurability is preserved by the function f.

Lemma 2. *Let x be a positive selfadjoint measurable operator. Then $x \log x$ is also measurable.*

Proof. Let x be a positive selfadjoint measurable operator, and let

$$x = \int_0^\infty t\, p(dt) \tag{7}$$

be its spectral representation. Then

$$x \log x = \int_0^\infty t \log t\, p(dt) = \int_0^1 t \log t\, p(dt) + \int_1^\infty t \log t\, p(dt).$$

The first integral on the right hand side of the equality above represents a bounded operator, so $\int_0^1 t \log t\, p(dt) \in \mathscr{M}$. For the second integral we have

$$\int_1^\infty t \log t\, p(dt) = \int_0^\infty \lambda\, (f \circ p)(d\lambda),$$

where the measure $f \circ p$ is defined as

$$(f \circ p)(E) = p(f^{-1}(E)), \quad E \in \mathcal{B}(\mathbf{R}).$$

The operator $\int_1^\infty t \log t\, p(dt)$ is selfadjoint and positive, and its spectral measure is $f \circ p$. Since x is measurable, there is $t_0 > 0$ such that $\tau(p([t_0, \infty))) < \infty$. Let λ_0 be such that $f(t_0) = \lambda_0$. Then we have

$$\tau((f \circ p)([\lambda_0, \infty))) = \tau(p(f^{-1}([\lambda_0, \infty)))) = \tau(p([t_0, \infty))) < \infty,$$

showing that $\int_1^\infty t \log t\, p(dt) = \int_0^\infty \lambda\, (f \circ p)(d\lambda)$ is measurable. It follows that $x \log x$ is measurable as a sum of two measurable operators. □

We have the following counterpart of a result known for bounded operators.

Theorem 3. *The function f defined by the formula* (6) *is strictly operator convex for positive selfadjoint measurable operators.*

Proof. The method of proof is similar to that for bounded operators, however, a number of refinements are required due to the unboundedness of the operators in question. Let x be a positive selfadjoint measurable operator with spectral representation (7). Take an arbitrary $\xi \in \mathcal{D}(x \log x)$. The representation

$$t \log t = \int_0^\infty \Big(\frac{t}{s+1} - \frac{t}{s+t}\Big) ds,$$

yields

$$\langle (x \log x)\xi|\xi\rangle = \int_0^\infty \Big(\int_0^\infty \Big(\frac{t}{s+1} - \frac{t}{s+t}\Big) ds\Big) \|p(dt)\xi\|^2.$$

The following estimates hold true for the function under the integral sign

$$\begin{aligned}
\Big|\frac{t}{s+1} - \frac{t}{s+t}\Big| &= \frac{|t^2 - t|}{(s+1)(s+t)} \leqslant 1, &&\text{for } 0 \leqslant s \leqslant 1,\ 0 \leqslant t \leqslant 1,\\
\Big|\frac{t}{s+1} - \frac{t}{s+t}\Big| &= \frac{|t^2 - t|}{(s+1)(s+t)} \leqslant t, &&\text{for } 0 \leqslant s \leqslant 1,\ t > 1,\\
\Big|\frac{t}{s+1} - \frac{t}{s+t}\Big| &= \frac{|t^2 - t|}{(s+1)(s+t)} \leqslant \frac{1}{s^2}, &&\text{for } s > 1,\ 0 \leqslant t \leqslant 1,\\
\Big|\frac{t}{s+1} - \frac{t}{s+t}\Big| &= \frac{|t^2 - t|}{(s+1)(s+t)} \leqslant \frac{t^2}{s^2}, &&\text{for } s > 1,\ t > 1,
\end{aligned}$$

consequently,

$$\begin{aligned}
&\iint\limits_{[0,1]\times[0,1]} \Big|\frac{t}{s+1} - \frac{t}{s+t}\Big| (ds \times \|p(dt)\xi\|^2)\\
&\leqslant \iint\limits_{[0,1]\times[0,1]} (ds \times \|p(dt)\xi\|^2) = \|p([0,1])\xi\|^2 \leqslant \|\xi\|^2 < \infty,\\
&\iint\limits_{[0,1]\times(1,\infty)} \Big|\frac{t}{s+1} - \frac{t}{s+t}\Big| (ds \times \|p(dt)\xi\|^2)\\
&\leqslant \iint\limits_{[0,1]\times(1,\infty)} t\,(ds \times \|p(dt)\xi\|^2) = \int\limits_{(1,\infty)} t\,\|p(dt)\xi\|^2\\
&\leqslant \int\limits_{(1,\infty)} t^2\,\|p(dt)\xi\|^2 \leqslant \int_0^\infty t^2\,\|p(dt)\xi\|^2 = \|x\xi\|^2 < \infty,\\
&\iint\limits_{(1,\infty)\times[0,1]} \Big|\frac{t}{s+1} - \frac{t}{s+t}\Big| (ds \times \|p(dt)\xi\|^2)\\
&\leqslant \iint\limits_{(1,\infty)\times[0,1])} \frac{1}{s^2}\,(ds \times \|p(dt)\xi\|^2) = \Big(\int_1^\infty \frac{1}{s^2}\,ds\Big) \|p([0,1])\xi\|^2\\
&\leqslant \|\xi\|^2 < \infty,
\end{aligned}$$

$$\begin{aligned}&\iint\limits_{(1,\infty)\times(1,\infty)} \Big|\frac{t}{s+1}-\frac{t}{s+t}\Big|\,(ds\times\|p(dt)\xi\|^2)\\ &\leqslant \iint\limits_{(1,\infty)\times(1,\infty)} \frac{t^2}{s^2}\,(ds\times\|p(dt)\xi\|^2)=\Big(\int_1^\infty \frac{1}{s^2}\,ds\Big)\Big(\int_1^\infty t^2\,\|p(dt)\xi\|^2\Big)\\ &\leqslant \int_0^\infty t^2\,\|p(dt)\xi\|^2=\|x\xi\|^2<\infty.\end{aligned}$$

It follows that the function $(s,t)\mapsto \frac{t}{s+1}-\frac{t}{s+t}$ is integrable, and the Fubini theorem yields

$$\begin{aligned}\langle (x\log x)\xi|\xi\rangle &= \int_0^\infty \Big(\int_0^\infty \Big(\frac{t}{s+1}-\frac{t}{s+t}\Big)ds\Big)\|p(dt)\xi\|^2\\ &= \int_0^\infty \Big(\int_0^\infty \Big(\frac{t}{s+1}-\frac{t}{s+t}\Big)\|p(dt)\xi\|^2\Big)ds\\ &= \int_0^\infty \Big\langle\Big(\frac{1}{s+1}x-x(s\mathbf{1}+x)^{-1}\Big)\xi|\xi\Big\rangle\,ds\\ &= \int_0^\infty \Big\langle\Big(\frac{1}{s+1}x-\mathbf{1}+s(s\mathbf{1}+x)^{-1}\Big)\xi|\xi\Big\rangle\,ds.\end{aligned}$$

For arbitrary positive selfadjoint measurable operators x_1, x_2 and $\lambda\in(0,1)$ the operators

$$\lambda x_1\log x_1+(1-\lambda)x_2\log x_2=\lambda f(x_1)+(1-\lambda)f(x_2)$$

and

$$(\lambda x_1+(1-\lambda)x_2)\log(\lambda x_1+(1-\lambda)x_2)=f(\lambda x_1+(1-\lambda)x_2)$$

are selfadjoint and measurable. Let ξ belong to the intersection of their domains. Then

$$\begin{aligned}&\langle(\lambda f(x_1)+(1-\lambda)f(x_2)-f(\lambda x_1+(1-\lambda)x_2))\xi|\xi\rangle\\ &=\int_0^\infty s\langle((s\mathbf{1}+\lambda x_1)^{-1}+\big(s\mathbf{1}+(1-\lambda)x_2\big)^{-1}+\\ &\quad-\big(s\mathbf{1}+\lambda x_1+(1-\lambda)x_2\big)^{-1})\xi|\xi\rangle\,ds\geqslant 0,\end{aligned}\tag{8}$$

by virtue of Proposition 1. Moreover, the equality

$$\lambda f(x_1)+(1-\lambda)f(x_2)=f(\lambda x_1+(1-\lambda)x_2)$$

yields that the term under the integral sign in (8) is zero for all ξ in the intersection of the respective domains, i.e. that

$$(s\mathbf{1}+\lambda x_1)^{-1}+\big(s\mathbf{1}+(1-\lambda)x_2\big)^{-1}=\big(s\mathbf{1}+\lambda x_1+(1-\lambda)x_2\big)^{-1}$$

on a dense subspace, hence everywhere, since the operators above are bounded, and again Proposition 1 yields the equality $x_1=x_2$. □

The point of the lemma that follows is that the function considered there has an integral representation with the integral over a bounded interval, while the integral representation of the function defined by the formula (6) deals with an integral over the interval $[0,\infty)$. This will be employed later for the Bochner integrability of some appropriate function.

Lemma 4. *Let m and M be such that $0 < m < M$. The function*

$$f_{m,M}(t) = t\log(m+t) - t\log(m+1) + t\log(M+1) - t\log(M+t), \quad t \geqslant 0, \tag{9}$$

is strictly operator convex on $\mathbf{B}(\mathcal{H})^+$.

Proof. The method of proof is essentially the same as that for proving the strict operator convexity of the function $t \mapsto t\log t$, and is based on the representation

$$f_{m,M}(t) = \int_m^M \Big(\frac{t}{s+1} - \frac{t}{s+t}\Big)\, ds \tag{10}$$

in the same way in which the operator convexity of the function $t \mapsto t\log t$ is based on the representation

$$t\log t = \int_0^\infty \Big(\frac{t}{s+1} - \frac{t}{s+t}\Big)\, ds,$$

so we omit the details. □

In what follows, while speaking about $0 \leqslant h \in L^1(\mathscr{M}, \tau)$ we shall assume that it has the spectral representation

$$h = \int_0^\infty t\, e(dt). \tag{11}$$

Let $0 \leqslant h \in L^1(\mathscr{M}, \tau)$. The operator $f_{m,M}(h)$ is defined by means of the spectral theorem, i.e.

$$f_{m,M}(h) = \int_0^\infty f_{m,M}(t)\, e(dt).$$

We shall prove that $f_{m,M}(h)$ has also another representation.

Lemma 5. *For $0 \leqslant h \in L^1(\mathscr{M}, \tau)$, we have*

$$f_{m,M}(h) = \int_m^M \Big(\frac{1}{s+1}h - h(s\mathbf{1}+h)^{-1}\Big)\, ds, \tag{12}$$

where the integral on the right hand side is Bochner's integral of a function with values in $L^1(\mathscr{M}, \tau)$.

Proof. Observe first that we have

$$\begin{aligned}\|h(s'\mathbf{1}+h)^{-1} - h(s''\mathbf{1}+h)^{-1}\|_1 &= |s'-s''|\,\|h(s''\mathbf{1}+h)^{-1}(s'\mathbf{1}+h)^{-1}\|_1 \\ &\leqslant |s'-s''|\,\|h\|_1\|(s''\mathbf{1}+h)^{-1}(s'\mathbf{1}+h)^{-1}\|_\infty \leqslant \frac{|s'-s''|}{s's''}\|h\|_1,\end{aligned}$$

which shows that the function

$$[m, M] \ni s \mapsto \frac{1}{s+1}h - h(s\mathbf{1}+h)^{-1}$$

is continuous in $\|\cdot\|_1$-norm, hence strongly measurable. Moreover,

$$\begin{aligned}\Big\|\frac{1}{s+1}h - h(s\mathbf{1}+h)^{-1}\Big\|_1 &= \frac{1}{s+1}\|h(h-\mathbf{1})(s\mathbf{1}+h)^{-1}\|_1 \\ &\leqslant \frac{1}{s+1}\|h\|_1\|(h-\mathbf{1})(s\mathbf{1}+h)^{-1}\|_\infty \leqslant \frac{\max\big\{1, \frac{1}{s}\big\}}{s+1}\|h\|_1,\end{aligned} \tag{13}$$

because

$$\|(h-\mathbf{1})(s\mathbf{1}+h)^{-1}\|_\infty = \max\Big\{1, \frac{1}{s}\Big\}.$$

Since the function

$$[m,M]\ni s\mapsto \frac{\max\left\{1,\frac{1}{s}\right\}}{s+1}\|h\|_1$$

is integrable, it follows that there exists the Bochner integral on the right hand side of the equality (12). Denote this integral by z:

$$z=\int_m^M\left(\frac{1}{s+1}h-h(s\mathbf{1}+h)^{-1}\right)ds\in L^1(\mathscr{M},\tau).$$

For the spectral representation of h as in (11), and for each fixed positive integer r put

$$e_r=e([0,r]).$$

Then he_r is bounded, and in the same way as in the proof of Lemma 5, we obtain

$$\begin{aligned}&\|he_r(s'\mathbf{1}+h)^{-1}-he_r(s''\mathbf{1}+h)^{-1}\|_\infty\\&=|s'-s''|\|he_r(s''\mathbf{1}+h)^{-1}(s'\mathbf{1}+h)^{-1}\|_\infty\\&\leqslant|s'-s''|\|he_r\|_\infty\|(s''\mathbf{1}+h)^{-1}(s'\mathbf{1}+h)^{-1}\|_\infty\leqslant\frac{r|s'-s''|}{s's''},\end{aligned}$$

and

$$\begin{aligned}&\left\|\frac{1}{s+1}he_r-he_r(s\mathbf{1}+h)^{-1}\right\|_\infty=\frac{1}{s+1}\|he_r(h-\mathbf{1})(s\mathbf{1}+h)^{-1}\|_\infty\\&\leqslant\frac{1}{s+1}\|he_r\|_\infty\|(h-\mathbf{1})(s\mathbf{1}+h)^{-1}\|_\infty\leqslant\frac{r\max\left\{1,\frac{1}{s}\right\}}{s+1},\end{aligned}$$

which shows that there exists the Bochner integral

$$\int_m^M\left(\frac{1}{s+1}he_r-he_r(s\mathbf{1}+h)^{-1}\right)ds\in\mathscr{M}.$$

Now put

$$s_k^{(n)}=m+k\frac{M-m}{n},\quad k=0,1,\dots,n,$$

and define simple functions $g_n\colon[m,M]\to L^1(\mathscr{M},\tau)$ by the formula

$$g_n(s)=\frac{1}{s_k^{(n)}+1}h-h(s_k^{(n)}\mathbf{1}+h)^{-1}\quad\text{for } s\in\left[s_k^{(n)},s_{k+1}^{(n)}\right),$$

i.e.

$$g_n(s)=\sum_{k=0}^{n-1}\left(\frac{1}{s_k^{(n)}+1}h-h(s_k^{(n)}\mathbf{1}+h)^{-1}\right)\chi_{\left[s_k^{(n)},s_{k+1}^{(n)}\right)}(s),$$

where χ_Δ stands for the characteristic function of the set $\Delta\subset\mathbf{R}$. We have

$$g_n(s)\longrightarrow\frac{1}{s+1}h-h(s\mathbf{1}+h)^{-1}\quad\text{in }\|\cdot\|_1\text{-norm},$$

so for the integral sums

$$\begin{aligned}S_n&=\sum_{k=0}^{n-1}\left(\frac{1}{s_k^{(n)}+1}h-h(s_k^{(n)}\mathbf{1}+h)^{-1}\right)\left(s_{k+1}^{(n)}-s_k^{(n)}\right)\\&=\sum_{k=0}^{n-1}\left(\frac{1}{s_k^{(n)}+1}h-h(s_k^{(n)}\mathbf{1}+h)^{-1}\right)\frac{M-m}{n}\end{aligned}$$

we get

$$S_n \longrightarrow \int_m^M \Big(\frac{1}{s+1}h - h(s\mathbf{1}+h)^{-1}\Big)\,ds = z \quad \text{in } \|\cdot\|_1\text{-norm.}$$

Similarly, for the simple functions $g_n e_r\colon [m,M] \to \mathscr{M}$ defined as

$$g_n(s)e_r = \sum_{k=0}^{n-1}\Big(\frac{1}{s_k^{(n)}+1}he_r - he_r(s_k^{(n)}\mathbf{1}+h)^{-1}\Big)\chi_{\big[s_k^{(n)},s_{k+1}^{(n)}\big)}(s),$$

we have

$$g_n(s) \longrightarrow \frac{1}{s+1}he_r - he_r(s\mathbf{1}+h)^{-1} \quad \text{in } \|\cdot\|_\infty\text{-norm.}$$

Hence for the integral sums $S_n e_r$ we get

$$S_n e_r \longrightarrow \int_m^M \Big(\frac{1}{s+1}he_r - he_r(s\mathbf{1}+h)^{-1}\Big)\,ds \quad \text{in } \|\cdot\|_\infty\text{-norm.}$$

Now

$$\|(S_n - z)e_r\|_1 \leqslant \|S_n - z\|_1\|e_r\|_\infty \leqslant \|S_n - z\|_1 \to 0,$$

which means that

$$S_n e_r \longrightarrow z e_r \quad \text{in } \|\cdot\|_1\text{-norm.}$$

Since

$$S_n e_r \longrightarrow \int_m^M \Big(\frac{1}{s+1}he_r - he_r(s\mathbf{1}+h)^{-1}\Big)\,ds \quad \text{in } \|\cdot\|_\infty\text{-norm,}$$

it follows that

$$ze_r = \int_m^M \Big(\frac{1}{s+1}he_r - he_r(s\mathbf{1}+h)^{-1}\Big)\,ds.$$

The same procedure holds for the function $e_r g_n$, thus $ze_r = e_r z$.

Take an arbitrary $\xi \in e_r(\mathcal{H})$. Then we have, using the representation (10),

$$\begin{aligned}
\langle z\xi|\xi\rangle &= \Big\langle\Big(\int_m^M \Big(\frac{1}{s+1}he_r - he_r(s\mathbf{1}+h)^{-1}\Big)\,ds\Big)\xi|\xi\Big\rangle \\
&= \int_m^M \Big\langle\Big(\frac{1}{s+1}h - h(s\mathbf{1}+h)^{-1}\Big)\xi|\xi\Big\rangle\,ds \\
&= \int_m^M \int_0^\infty \Big(\frac{t}{s+1} - \frac{t}{s+t}\Big)\,\|e(dt)\xi\|^2 ds \\
&= \int_0^\infty \int_m^M \Big(\frac{t}{s+1} - \frac{t}{s+t}\Big)\,ds\|e(dt)\xi\|^2 \\
&= \int_0^\infty f_{m,M}(t)\|e(dt)\xi\|^2 = \langle f_{m,M}(h)\xi|\xi\rangle,
\end{aligned}$$

where the change of the order of integration is justified exactly as in the proof of Theorem 3. This shows that $z\xi = f_{m,M}(h)\xi$, i.e.

$$ze_r = f_{m,M}(h)e_r.$$

Since $\tau(e_r^\perp) \to 0$ as $r \to \infty$, we infer on account of [5, Theorem 2] (alternatively, one may use [8, Corollary 5.1]) that

$$z = f_{m,M}(h). \qquad \square$$

Now we want to show that in some important case the operator $h\log h$ can be approximated by $f_{m,M}(h)$.

Lemma 6. *Let $0 \leqslant h \in L^1(\mathscr{M},\tau)$ be a density with finite Segal's entropy, i.e. $h\log h \in L^1(\mathscr{M},\tau)$. Then*

$$\lim_{\substack{m\to 0\\ M\to\infty}} f_{m,M}(h) = h\log h \quad \textit{in } \|\cdot\|_1\textit{-norm.}$$

Proof. Let h have the spectral representation (11). We have the following equality

$$\|h\log(m\mathbf{1}+h) - h\log h\|_1 = \int_0^\infty (t\log(m+t) - t\log t)\,\tau(e(dt)),$$

and, assuming $m \leqslant 1$, the following estimate holds

$$t\log(m+t) \leqslant t\log(1+t) \leqslant \begin{cases} t\log 2, & \text{for } t\leqslant 1\\ t\log 2t, & \text{for } t>1\end{cases}$$
$$= \begin{cases} t\log 2, & \text{for } t\leqslant 1\\ t\log 2 + t\log t, & \text{for } t>1\end{cases} \leqslant t\log 2 + |t\log t|.$$

Since

$$t\log t \leqslant t\log(m+t),$$

we have

$$|t\log(m+t)| \leqslant \max\{|t\log t|, |t\log 2 + |t\log t||\} = t\log 2 + |t\log t|,$$

so the function under the integral sign is estimated by

$$|t\log(m+t) - t\log t| \leqslant t\log 2 + 2|t\log t|.$$

The finiteness of Segal's entropy of h means that $h\log h\in L^1(\mathscr{M},\tau)$, so

$$\int_0^\infty |t\log t|\,\tau(e(dt)) = \tau(|h\log h|) < \infty,$$

and of course

$$\int_0^\infty t\log 2\,\tau(e(dt)) = (\log 2)\tau(h) < \infty,$$

thus the function $t\mapsto t\log 2 + 2|t\log t|$ is integrable. Since

$$t\log(m+t) - t\log t \longrightarrow 0 \quad \text{as } m\to 0,$$

we get, using the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}&\lim_{m\to 0}\|h\log(m\mathbf{1}+h) - h\log h\|_1\\ &= \lim_{m\to 0}\int_0^\infty (t\log(m+t) - t\log t)\,\tau(e(dt))\\ &= \int_0^\infty \lim_{m\to 0}(t\log(m+t) - t\log t)\,\tau(e(dt)) = 0.\end{aligned}$$

Further, we have

$$\begin{aligned}&\|\log(M+1)h - h\log(M\mathbf{1}+h)\|_1 = \int_0^\infty t\left|\log\frac{M+1}{M+t}\right|\tau(e(dt))\\ &= \int_0^1 t\log\frac{M+1}{M+t}\,\tau(e(dt)) + \int_1^\infty t\log\frac{M+t}{M+1}\,\tau(e(dt)) \qquad (14)\\ &\leqslant \log\frac{M+1}{M}\int_0^1 t\,\tau(e(dt)) + \int_1^\infty t\log\frac{M+t}{M+1}\,\tau(e(dt)).\end{aligned}$$

For the functions

$$k_M(t) = t \log \frac{M+t}{M+1}, \quad t \in [1, \infty),$$

we have

$$\lim_{M\to\infty} k_M(t) = 0,$$

and

$$|k_M(t)| \leqslant t \log t,$$

which means that the functions k_M are bounded above by an integrable function, and passing to the limit in the inequality (14), we get, again on account of the Lebesgue Dominated Convergence Theorem,

$$\lim_{M\to\infty} \| \log(M+1)h - h\log(M\mathbf{1}+h)\|_1 = 0.$$

Consequently,

$$\begin{aligned}\|f_{m,M}(h) - h\log h\|_1 \leqslant \|h\log(m\mathbf{1}+h) - h\log h\|_1 + \| \log(m+1)h\|_1 \\ + \| \log(M+1)h - h\log(M\mathbf{1}+h)\|_1,\end{aligned}$$

and taking into account the obvious relation $\| \log(m+1)h\|_1 \underset{m\to 0}{\longrightarrow} 0$ we obtain the conclusion. □

For a positive selfadjoint measurable operator x with spectral representation

$$x = \int_0^\infty t\, p(dt)$$

denote by $x_{[n]}$ its restriction

$$x_{[n]} = \int_0^n t\, p(dt).$$

Let $0 \leqslant h \in L^1(\mathscr{M}, \tau)$ have finite entropy. Then

$$h\log h - h_{[n]}\log h_{[n]} = \int_n^\infty t\log t\, e(dt),$$

and thus

$$\|h\log h - h_{[n]}\log h_{[n]}\|_1 = \int_n^\infty t\log t\, \tau(e(dt)) \to 0, \tag{15}$$

since

$$H(h) = \tau(h\log h) = \int_0^\infty t\log t\, \tau(e(dt))$$

is finite. In an analogous way, we obtain for arbitrary $M > 0$

$$\begin{aligned}&\|h\log(M\mathbf{1}+h) - h_{[n]}\log(M\mathbf{1}+h_{[n]})\|_1 \\ &= \int_n^\infty t\log(M+t)\tau(e(dt)) \leqslant \int_n^\infty t\log 2t\, \tau(e(dt)) \\ &= \log 2\int_n^\infty t\, \tau(e(dt)) + \int_n^\infty t\log t\, \tau(e(dt)) \to 0,\end{aligned} \tag{16}$$

since

$$\int_0^\infty t\, \tau(e(dt)) = \tau(h) < \infty.$$

We have the following special, yet interesting in their own right, cases of the Jensen inequality for unbounded operators.

Theorem 7. *Let $\alpha\colon \mathscr{M} \to \mathscr{M}$ be a positive unital linear map having an extension to $L^1(\mathscr{M},\tau)$ bounded with respect to $\|\cdot\|_1$-norm, and let $0 \leqslant h \in L^1(\mathscr{M},\tau)$. Then*

(i) *for h having finite Segal's entropy,*

$$\alpha(h\log h) \geqslant \alpha(h)\log\alpha(h);$$

(ii) *for α normal,*

$$\alpha\big((s\mathbf{1}+h)^{-1}\big) \geqslant \big(s\mathbf{1}+\alpha(h)\big)^{-1}$$

for each $s>0$.

Proof. (i) Let the function $f_{m,M}$ be defined by the formula (9). Since this function is operator convex, the Jensen inequality for bounded operators yields

$$\alpha(f_{m,M}(h_{[n]})) \geqslant f_{m,M}\big(\alpha(h_{[n]})\big). \tag{17}$$

Taking into account the relations (15) and (16), we get

$$\begin{aligned} f_{m,M}(h_{[n]}) &= h_{[n]}\log(m\mathbf{1}+h_{[n]}) - \log(m+1)h_{[n]} + \log(M+1)h_{[n]} \\ &\quad - h_{[n]}\log(M\mathbf{1}+h_{[n]}) \longrightarrow h\log(m\mathbf{1}+h) - \log(m+1)h \\ &\quad + \log(M+1)h - h\log(M\mathbf{1}+h) = f_{m,M}(h), \end{aligned}$$

where the convergence is in $\|\cdot\|_1$-norm.

For the right-hand side of the inequality (17), we have

$$f_{m,M}\big(\alpha(h_{[n]})\big) = \int_m^M \Big(\frac{1}{s+1}\alpha(h_{[n]}) - \alpha(h_{[n]})(s\mathbf{1}+\alpha(h_{[n]}))^{-1}\Big)\,ds,$$

and putting $x=\alpha(h_{[n]})$ in the inequality (13), we get

$$\begin{aligned} &\Big\|\frac{1}{s+1}\alpha(h_{[n]}) - \alpha(h_{[n]})(s\mathbf{1}+\alpha(h_{[n]}))^{-1}\Big\|_1 \leqslant \frac{\max\{1,\frac{1}{s}\}}{s+1}\|\alpha(h_{[n]})\|_1 \\ &\leqslant \frac{\max\{1,\frac{1}{s}\}}{s+1}\|\alpha\|_1\|h_{[n]}\|_1 \leqslant \frac{\max\{1,\frac{1}{s}\}}{s+1}\|\alpha\|_1, \end{aligned}$$

which means that the sequence of functions under the integral sign is bounded above by an integrable function. Moreover, from the continuity of the function

$$[m,M] \ni s \mapsto \frac{1}{s+1}x - x(s\mathbf{1}+x)^{-1}$$

in $\|\cdot\|_1$-norm noticed at the beginning of the proof, it follows that

$$\frac{1}{s+1}\alpha(h_{[n]}) - \alpha(h_{[n]})(s\mathbf{1}+\alpha(h_{[n]}))^{-1} \to \frac{1}{s+1}\alpha(h) - \alpha(h)(s\mathbf{1}+\alpha(h))^{-1},$$

where the convergence is in $\|\cdot\|_1$-norm, thus on account of the Lebesgue Dominated Convergence Theorem for Bochner integrals, we obtain

$$\begin{aligned} f_{m,M}\big(\alpha(h_{[n]})\big) &= \int_m^M \Big(\frac{1}{s+1}\alpha(h_{[n]}) - \alpha(h_{[n]})(s\mathbf{1}+\alpha(h_{[n]}))^{-1}\Big)\,ds \\ &\longrightarrow \int_m^M \Big(\frac{1}{s+1}\alpha(h) - \alpha(h)(s\mathbf{1}+\alpha(h))^{-1}\Big)\,ds = f_{m,M}(\alpha(h)), \end{aligned}$$

again with the $\|\cdot\|_1$-convergence. Passing to the limit with $n\to\infty$ in $\|\cdot\|_1$-norm in the inequality (17), we get

$$\alpha(f_{m,M}(h)) \geqslant f_{m,M}\big(\alpha(h)\big).$$

Now passing to the limit with $m \to 0$ and $M \to \infty$ in the inequality above, we obtain by virtue of Lemma 6

$$\alpha(h \log h) \geqslant \alpha(h) \log \alpha(h),$$

which finishes the proof.

(ii) Let $0 \leqslant h_n \in \mathscr{M} \cap L^1(\mathscr{M}, \tau)$ be such that $h_n \to h$ in $\|\cdot\|_1$-norm. The Jensen inequality for bounded operators yields

$$\alpha\big((s\mathbf{1} + h_n)^{-1}\big) \geqslant \big(s\mathbf{1} + \alpha(h_n)\big)^{-1}. \tag{18}$$

We have

$$\begin{aligned}\|(s\mathbf{1} + h_n)^{-1} - (s\mathbf{1} + h)^{-1}\|_1 &= \|(s\mathbf{1} + h_n)^{-1}(h - h_n)(s\mathbf{1} + h)^{-1}\|_1 \\ &\leqslant \|(s\mathbf{1} + h_n)^{-1}\|_\infty \|h - h_n\|_1 \|(s\mathbf{1} + h)^{-1}\|_\infty \leqslant \frac{1}{s^2}\|h - h_n\|_1.\end{aligned}$$

Take an arbitrary $y \in \mathscr{M} \cap L^1(\mathscr{M}, \tau)$. Then we have

$$\begin{aligned}\big|(y\tau)\big((s\mathbf{1} + h_n)^{-1} - (s\mathbf{1} + h)^{-1}\big)\big| &= \big|\tau\big(\big((s\mathbf{1} + h_n)^{-1} - (s\mathbf{1} + h)^{-1}\big)y\big)\big| \\ &\leqslant \|(s\mathbf{1} + h_n)^{-1} - (s\mathbf{1} + h)^{-1}\|_1 \|y\|_\infty \leqslant \frac{1}{s^2}\|y\|_\infty \|h - h_n\|_1 \to 0,\end{aligned}$$

which means that

$$\varphi\big((s\mathbf{1} + h_n)^{-1}\big) \to \varphi\big((s\mathbf{1} + h)^{-1}\big)$$

for all $\varphi \in \mathscr{M}_*$ of the form $\varphi = y\tau$ where $y \in \mathscr{M} \cap L^1(\mathscr{M}, \tau)$. Since such φ are dense in norm in $\mathscr{M}_*$, and $(s\mathbf{1} + h_n)^{-1}$ are bounded in $\|\cdot\|_\infty$-norm, it follows that

$$\varphi\big((s\mathbf{1} + h_n)^{-1}\big) \to \varphi\big((s\mathbf{1} + h)^{-1}\big)$$

for all $\varphi \in \mathscr{M}_*$, i.e.

$$(s\mathbf{1} + h_n)^{-1} \to (s\mathbf{1} + h)^{-1} \quad \sigma\text{-weakly.}$$

Since $0 \leqslant \alpha(h_n) \in \mathscr{M} \cap L^1(\mathscr{M}, \tau)$ and $\alpha(h_n) \to \alpha(h)$ in $\|\cdot\|_1$-norm, we get by the same token

$$(s\mathbf{1} + \alpha(h_n))^{-1} \to (s\mathbf{1} + \alpha(h))^{-1} \quad \sigma\text{-weakly,}$$

and since α is normal, passing to the limit in the inequality (18) gives the claim. □

The next lemma shows the possibility of extension to $L^1(\mathscr{M}, \tau)$ of a map with respect to which the trace is invariant. A similar (much deeper) result for conditional expectation shows that such an extension does not increase the $\|\cdot\|_1$-norm, however, we need only the boundedness of this extension.

Lemma 8. *Let α be a positive linear unital map on $\mathscr{M}$ such that $\tau \circ \alpha = \tau$. Then α can be extended to a bounded linear map on $L^1(\mathscr{M}, \tau)$ such that $\|\alpha\|_1 \leqslant 2$.*

Proof. Take an arbitrary $x \in \mathscr{M}^h \cap L^1(\mathscr{M}, \tau)$. Then $x = x^+ - x^-$ where $x^+, x^- \in \mathscr{M}^+ \cap L^1(\mathscr{M}, \tau)$, and we have

$$\begin{aligned}\|\alpha(x)\|_1 \leqslant \|\alpha(x^+\|_1 + \|\alpha(x^-)\|_1 &= \tau(\alpha(x^+)) + \tau(\alpha(x^-)) \\ &= \tau(x^+) + \tau(x^-) = \tau(x^+ + x^-) = \tau(|x|) = \|x\|_1,\end{aligned}$$

which means that α can be extended to a map on $L^1(\mathscr{M}, \tau)^h$ with norm one (of course, we have for $x \in \mathscr{M}^+ \cap L^1(\mathscr{M}, \tau)$, $\|\alpha(x)\|_1 = \tau(\alpha(x)) = \tau(x) = \|x\|_1$, showing that $\|\alpha\|_1 = 1$). For arbitrary $x \in L^1(\mathscr{M}, \tau)$, we have $x = x_1 + ix_2$, where

$$x_1 = \frac{x + x^*}{2} \in L^1(\mathscr{M}, \tau)^h, \quad x_2 = \frac{x - x^*}{2i} \in L^1(\mathscr{M}, \tau)^h,$$

and putting
$$\alpha(x) = \alpha(x_1) + i\alpha(x_2),$$
we get
$$\|\alpha(x)\|_1 \leqslant \|\alpha(x_1)\|_1 + \|\alpha(x_2)\|_1 \leqslant \|x_1\|_1 + \|x_2\|_1 \leqslant 2\|x\|_1,$$
since
$$\|x_1\|_1 = \left\|\frac{x + x^*}{2}\right\|_1 \leqslant \frac{\|x\|_1 + \|x^*\|_1}{2} = \|x\|_1,$$
and
$$\|x_2\|_1 = \left\|\frac{x - x^*}{2i}\right\|_1 \leqslant \frac{\|x\|_1 + \|x^*\|_1}{2} = \|x\|_1. \qquad \square$$

Let α be a positive normal linear unital map on $\mathscr{M}$ such that $\tau \circ \alpha = \tau$. Then there exists a positive normal linear unital map $\widetilde{\alpha}$ on $\mathscr{M}$, called *conjugate to* α, such that $\tau \circ \widetilde{\alpha} = \tau$. The map $\widetilde{\alpha}$ is defined by the formula
$$\tau(x\widetilde{\alpha}(y)) = \tau(\alpha(x)y), \quad x, y \in \mathscr{M} \cap L^1(\mathscr{M}, \tau).$$
This map is faithful. Indeed, if $\widetilde{\alpha}(x) = 0$ for $x \geqslant 0$, then we have
$$0 = \tau(\widetilde{\alpha}(x)) = \tau(x),$$
and the faithfulness of τ yields $x = 0$. (Of course, the same reasoning shows that α is faithful too.) The existence of the map $\widetilde{\alpha}$ was mentioned in [7], while its detailed construction together with proving its properties was performed in [4].

3. Mappings preserving entropy (general case)

In this section, we want to characterise the situation when a normal positive linear unital map on a semifinite von Neumann algebra does not change the entropy of a density.

For arbitrary elements $x, y \in \mathbf{B}(\mathcal{H})$, we define their Jordan product $x \circ y$ as
$$x \circ y = \frac{xy + yx}{2}.$$
The next propositions are interesting in their own right.

Proposition 9. *Let $\mathscr{M}$ be an arbitrary von Neumann algebra, and let $\alpha\colon \mathscr{M} \to \mathscr{M}$ be a positive linear contraction. Let $x \in \mathscr{M}$ be such that*
$$\alpha(x^* \circ x) = \alpha(x)^* \circ \alpha(x). \tag{19}$$
Then for arbitrary $y \in \mathscr{M}$ we have
$$\alpha(x \circ y) = \alpha(x) \circ \alpha(y).$$

Proof. For each $z \in \mathscr{M}$ the following version of the Schwarz inequality holds
$$\alpha(z^* \circ z) \geqslant \alpha(z)^* \circ \alpha(z).$$
Take an arbitrary positive linear functional φ on $\mathscr{M}$ and define a sesquilinear form $[\cdot,\cdot]_\varphi$ on $\mathscr{M} \times \mathscr{M}$ by the formula
$$[z, y]_\varphi = \varphi(\alpha(z \circ y^*) - \alpha(z) \circ \alpha(y)^*).$$
From the Schwarz inequality it follows that this form is positive, consequently, we have for arbitrary $y \in \mathscr{M}$
$$|[x, y]_\varphi|^2 \leqslant [x, x]_\varphi [y, y]_\varphi,$$
and the relation (19) gives $[x, x]_\varphi = 0$. It follows that
$$0 = [x, y]_\varphi = \varphi(\alpha(x \circ y^*) - \alpha(x) \circ \alpha(y)^*),$$

and since φ was arbitrary, we obtain the equality

$$\alpha(x \circ y^*) = \alpha(x) \circ \alpha(y)^*.$$

Now taking y^* instead of y shows the claim. □

Proposition 10. *Let $\mathscr{M}$ be an arbitrary von Neumann algebra, and let $\alpha\colon \mathscr{M} \to \mathscr{M}$ be a positive linear contraction. Let $z^* = z \in \mathscr{M}$ be such that*

$$\alpha\big(z^2\big) = \alpha(z)^2.$$

Then α restricted to the C-algebra $C^*(z)$ generated by z is a *-homomorphism. Moreover, if α is normal, then α restricted to the von Neumann algebra $W^*(z)$ generated by z is a *-homomorphism.*

Proof. From Proposition 9 we get

$$\alpha\big(z^3\big) = \alpha\big(z \circ z^2\big) = \alpha(z) \circ \alpha\big(z^2\big) = \alpha(z) \circ \alpha(z)^2 = \alpha(z)^3,$$

and by induction

$$\alpha(z^n) = \alpha(z)^n$$

for arbitrary positive integer n. From this equality, it follows that for every polynomial W we have

$$\alpha(W(z)) = W(\alpha(z)),$$

consequently, for every continuous function f on the spectrum of z the following equality holds

$$\alpha(f(z)) = f(\alpha(z)).$$

Since such functions form the C*-algebra $C^*(z)$ generated by z, for each $x, y \in C^*(z)$ there are continuous functions f and g such that $x = f(z)$, $y = g(z)$, and we get

$$\begin{aligned}\alpha(xy) &= \alpha(f(z)g(z)) = \alpha((fg)(z)) = (fg)(\alpha(z))\\ &= f(\alpha(z))g(\alpha(z)) = \alpha(f(z))\alpha(g(z)) = \alpha(x)\alpha(y)\end{aligned}$$

showing that α is a homomorphism on $C^*(z)$. Let now $y \in C^*(z)$ be arbitrary, and let $x \in W^*(z)$. On account of the Kaplansky density theorem, there is a net $\{x_i\}$ in $C^*(z)$ such that $\|x_i\| \leqslant \|x\|$ and $x_i \to x$ *σ-weakly*. Then $x_i y \to xy$ *σ-weakly*, and since α is normal we have $\alpha(x_i) \to \alpha(x)$ *σ-weakly* and $\alpha(x_i y) \to \alpha(xy)$ *σ-weakly*. Consequently, we obtain

$$\alpha(xy) = \lim_i \alpha(x_i y) = \lim_i \alpha(x_i)\alpha(y) = \alpha(x)\alpha(y)$$

for $x \in W^*(z)$, $y \in C^*(z)$. Repeating the reasoning above for arbitrary $x \in W^*(z)$ and $y \in W^*(z)$ (approximating y by elements in $C^*(z)$), we show that $\alpha|W^*(z)$ is a homomorphism. □

The subinvariance property of Segal's entropy, which is the content of the next theorem, is an immediate corollary to the Jensen inequality for the function $h \mapsto h \log h$, obtained in full generality in Theorem 7 (i). This result with the assumption that the corresponding density belongs to the algebra was proved in [7, Proposition 7.3] also by means of Jensen's inequality but for bounded operators.

Remark. The definition of Segal's entropy bears a strong resemblance to the classical Boltzmann–Gibbs notion, where for a density function f on a probability space $(\Omega, \mathfrak{F}, \mu)$ its entropy is defined as

$$H(f) = \int_\Omega f \log f \, d\mu.$$

It should be noted that the original Segal definition differs from ours by a minus sign before the trace. However, for simplicity and the sake of having nonnegative entropy for normalised states on a finite von Neumann algebra we have adopted the 'plus-sign' version.

Theorem 11. *Let $\mathscr{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\alpha\colon \mathscr{M} \to \mathscr{M}$ be a normal positive linear unital map such that $\tau \circ \alpha = \tau$. For arbitrary $0 \leqslant h \in L^1(\mathscr{M}, \tau)$ with finite entropy we have*

$$H(\alpha(h)) \leqslant H(h).$$

Proof. On account of Theorem 7 (i) we have

$$\alpha(h) \log \alpha(h) \leqslant \alpha(h \log h),$$

and applying τ to both sides of the inequality above yields the claim. □

Now we are in a position to prove one of the main results of the paper concerning the invariance of Segal's entropy under the action of a normal positive linear unital map. This result generalises that of Choda [1] obtained for the von Neumann entropy in finite dimension. It should be noted that in that case one simply refers to the finite discrete spectral representation of the density matrix. Obviously, nothing of this kind takes place for arbitrary semifinite von Neumann algebras.

Theorem 12. *Let $\mathscr{M}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ, and let $\alpha\colon \mathscr{M} \to \mathscr{M}$ be a normal positive linear unital map such that $\tau \circ \alpha = \tau$. For arbitrary $0 \leqslant h \in L^1(\mathscr{M}, \tau)$ with finite entropy the following conditions are equivalent:*

(i) $H(h) = H(\alpha(h))$,

(ii) *the map α restricted to the von Neumann algebra $W^*(h)$ generated by h is a *-isomorphism,*

(iii) $\widetilde{\alpha}(\alpha(h)) = h$.

Proof. (i) $\implies$ (ii) Assume that $H(h) = H(\alpha(h))$. From the α-invariance of τ, it follows that

$$\tau(\alpha(h \log h)) = \tau(h \log h) = H(h) = H(\alpha(h)) = \tau(\alpha(h) \log \alpha(h)).$$

On account of Theorem 7 (i), we have

$$\alpha(h \log h) \geqslant \alpha(h) \log \alpha(h),$$

and the equality of entropies yields

$$\tau(\alpha(h \log h) - \alpha(h) \log \alpha(h)) = 0.$$

The faithfulness of τ gives the equality

$$\alpha(h \log h) = \alpha(h) \log \alpha(h).$$

For the function $f_{m,M}$ defined by the formula (9), we have, by virtue of Lemma 5 and the equality

$$h(s\mathbf{1} + h)^{-1} = \mathbf{1} - s(s\mathbf{1} + h)^{-1},$$

the relation

$$\begin{aligned}\alpha(f_{m,M}(h)) &= \alpha\Big(\int_m^M \Big(\frac{1}{s+1} h - h(s\mathbf{1} + h)^{-1} \Big)\, ds \Big) \\ &= \int_m^M \Big(\frac{1}{s+1} \alpha(h) - \mathbf{1} + s\alpha\big((s\mathbf{1} + h)^{-1}\big) \Big)\, ds,\end{aligned}$$

and similarly,

$$f_{m,M}(\alpha(h)) = \int_m^M \Big(\frac{1}{s+1}\alpha(h) - \mathbf{1} + s\big((s\mathbf{1}+\alpha(h))^{-1}\big)\Big)\, ds.$$

Consequently,

$$\alpha(f_{m,M}(h)) - f_{m,M}(\alpha(h)) = \int_m^M s\big(\alpha\big((s\mathbf{1}+h)^{-1}\big) - (s\mathbf{1}+\alpha(h))^{-1}\big)\, ds.$$

From Theorem 7 (ii), it follows that the difference of the operators under the integral sign is nonnegative, so $\alpha(f_{m,M}(h)) - f_{m,M}(\alpha(h))$ is nonnegative and gets bigger as $m \searrow 0$ and $M \nearrow \infty$. This together with Lemma 6 yields

$$0 \leqslant \alpha(f_{m,M}(h)) - f_{m,M}(\alpha(h)) \nearrow \alpha(h\log h) - \alpha(h)\log\alpha(h) = 0$$

as $m \searrow 0$ and $M \nearrow \infty$, showing that

$$\alpha(f_{m,M}(h)) - f_{m,M}(\alpha(h)) = 0.$$

Consequently,

$$\alpha\big((s\mathbf{1}+h)^{-1}\big) - (s\mathbf{1}+\alpha(h))^{-1} = 0$$

almost everywhere in the interval $[m, M]$, and thus everywhere since the functions $s \mapsto \alpha\big(s\mathbf{1}+h)^{-1}\big)$ and $s \mapsto \big(s\mathbf{1}+\alpha(h)\big)^{-1}$ are continuous in $\|\cdot\|_\infty$-norm. More than that, these functions are differentiable because we have

$$\begin{aligned}&\frac{(s\mathbf{1}+h)^{-1} - (s_0\mathbf{1}+h)^{-1}}{s-s_0} \\ &= -(s\mathbf{1}+h)^{-1}(s_0\mathbf{1}+h)^{-1} \underset{s\to s_0}{\longrightarrow} -(s_0\mathbf{1}+h)^{-2} \quad \text{in } \|\cdot\|_\infty\text{-norm.}\end{aligned}$$

Taking derivatives in the equality

$$\alpha\big((s\mathbf{1}+h)^{-1}\big) = (s\mathbf{1}+\alpha(h))^{-1},$$

we obtain

$$\alpha\big((s\mathbf{1}+h)^{-2}\big) = (s\mathbf{1}+\alpha(h))^{-2}.$$

Put $z = (s\mathbf{1}+h)^{-1} \in \mathscr{M}$ for some fixed s. Then we have

$$\begin{aligned}\alpha(z^2) &= \alpha\big((s\mathbf{1}+h)^{-2}\big) = (s\mathbf{1}+\alpha(h))^{-2} \\ &= \big((s\mathbf{1}+\alpha(h)^{-1}\big)^2 = \big(\alpha\big((s\mathbf{1}+h)^{-1}\big)\big)^2 = \alpha(z)^2.\end{aligned}$$

From Proposition 10, it follows that α restricted to the von Neumann algebra $W^*\big((s\mathbf{1}+h)^{-1}\big)$ generated by $(s\mathbf{1}+h)^{-1}$ is a *-homomorphism, and being faithful it is an isomorphism. Since we have the equality $W^*\big((s\mathbf{1}+h)^{-1}\big) = W^*(h)$, the conclusion follows.

(ii) $\Longrightarrow$ (i) The only thing that must be taken care of is the fact that h may be unbounded. However, we have

$$\alpha\big(h_{[n]}\big) = \alpha\Big(\int_0^n t\, e(dt)\Big) = \int_0^n t\,\alpha(e(dt)),$$

and $\alpha(e(\cdot))$ is a spectral measure. Consequently, for the operator

$$x = \int_0^\infty t\,\alpha(e(dt))$$

we have

$$\tau(x) = \tau\Big(\int_0^\infty t\,\alpha(e(dt))\Big) = \int_0^\infty t\,\tau(\alpha(e(dt))) = \int_0^\infty t\,\tau(e(dt)) = \tau(h) < +\infty,$$

which means that $x \in L^1(\mathscr{M},\tau)$. Moreover,

$$\|x-\alpha\big(h_{[n]}\big)\|_1=\tau\Big(\int_n^\infty t\,\alpha(e(dt))\Big)=\int_n^\infty t\,\tau(\alpha(e(dt)))$$
$$=\int_n^\infty t\,\tau(e(dt))=\|h-h_{[n]}\|_1\to 0.$$

Since $\alpha\big(h_{[n]}\big)\to\alpha(h)$ in $\|\cdot\|_1$-norm, we obtain

$$\alpha(h)=x=\int_0^\infty t\,\alpha(e(dt)).$$

In particular,

$$\alpha(h)_{[n]}=\int_0^n t\,\alpha(e(dt))=\alpha\Big(\int_0^n t\,e(dt)\Big)=\alpha(h_{[n]}).$$

For every $z\in W^*(h)$ and every continuous function f, we have, since α is a *-isomorphism,

$$\alpha(f(z))=f(\alpha(z)),$$

thus

$$\alpha\big(h_{[n]}\log h_{[n]}\big)=\alpha\big(h_{[n]}\big)\log\alpha\big(h_{[n]}\big)=\alpha(h)_{[n]}\log\big(\alpha(h)_{[n]}\big),$$

and passing to the limit in $\|\cdot\|_1$-norm we get the equality

$$\alpha(h\log h)=\alpha(h)\log\alpha(h),$$

which gives the relation $H(h)=H(\alpha(h))$.

(ii) $\implies$ (iii) For arbitrary $x\in W^*(h)$, we have

$$\alpha(x^*\circ x)=\alpha(x)^*\circ\alpha(x),$$

thus according to Proposition 9, for any $y\in\mathscr{M}$ the relation

$$\alpha(x\circ y)=\alpha(x)\circ\alpha(y)$$

holds. Consequently, taking arbitrary $x,y\in\mathscr{M}\cap L^1(\mathscr{M},\tau)$ we obtain

$$\tau(\widetilde{\alpha}(\alpha(x))\circ y)=\tau(\alpha(x)\circ\alpha(y))=\tau(\alpha(x\circ y))=\tau(x\circ y),$$

i.e.

$$\tau((\widetilde{\alpha}(\alpha(x))-x)\circ y)=0.$$

Putting $y=(\widetilde{\alpha}(\alpha(x))-x)^*$ we get

$$\tau((\widetilde{\alpha}(\alpha(x))-x)\circ(\widetilde{\alpha}(\alpha(x))-x)^*)=0,$$

and the faithfulness of τ yields

$$\widetilde{\alpha}(\alpha(x))=x.$$

Now choose $h_n\in\mathscr{M}\cap L^1(\mathscr{M},\tau)$ such that $h_n\to h$ in $\|\cdot\|_1$-norm. Then

$$\widetilde{\alpha}(\alpha(h_n))=h_n,$$

and passing to the limit gives the claim. (Observe that in the course of proof we have actually shown that $\widetilde{\alpha}\circ\alpha$ is the identity map on $W^*(h)$ and that this statement is true also for the space $L^1(W^*(h),\tau|W^*(h))$.)

(iii) $\implies$ (ii) Since

$$\frac{t}{1+t}=1-\frac{1}{1+t},$$

we infer on account of Proposition 1 that the function $t \mapsto \frac{t}{1+t}$ is strictly operator concave on $\widetilde{\mathscr{M}}^+$. Put $\Phi = \widetilde{\alpha} \circ \alpha$. Then Φ is a normal positive linear unital map such that $\tau \circ \Phi = \tau$, and the above-mentioned operator concavity yields

$$\Phi\big(h(\mathbf{1}+h)^{-1}\big) \leqslant \Phi(h)\big(\mathbf{1}+\Phi(h)\big)^{-1} = h(\mathbf{1}+h)^{-1}.$$

Since $h(\mathbf{1}+h)^{-1} \in \mathscr{M} \cap L^1(\mathscr{M}, \tau)$, we get

$$\tau\big(h(\mathbf{1}+h)^{-1} - \Phi\big(h(\mathbf{1}+h)^{-1}\big)\big) = 0,$$

so the faithfulness of τ gives

$$\Phi\big(h(\mathbf{1}+h)^{-1}\big) = h(\mathbf{1}+h)^{-1}.$$

Put $z = h(\mathbf{1}+h)^{-1}$. Then $z \in \mathscr{M} \cap L^1(\mathscr{M}, \tau)$ and

$$\widetilde{\alpha}(\alpha(z)) = \Phi(z) = z.$$

Furthermore, we have $z^2 \in \mathscr{M} \cap L^1(\mathscr{M}, \tau)$ as a product of an element from $\mathscr{M}$ by an element from $L^1(\mathscr{M}, \tau)$, and

$$\tau\big(z^2\big) = \tau\big(\alpha\big(z^2\big)\big) \geqslant \tau\big(\alpha(z)^2\big) = \tau(\alpha(z)\alpha(z)) = \tau(\widetilde{\alpha}(\alpha(z))z) = \tau(z^2),$$

which gives

$$\tau\big(\alpha\big(z^2\big) - \alpha(z)^2\big) = 0.$$

Since

$$\alpha\big(z^2\big) \geqslant \alpha(z)^2,$$

the faithfulness of τ yields

$$\alpha\big(z^2\big) = \alpha(z)^2.$$

On account of Proposition 10, we infer that α is a *-homomorphism on $W^*\big((h(\mathbf{1}+h)^{-1}\big) = W^*(h)$, and being faithful it is a *-isomorphism which finishes the proof. □

Remark. For better clarity, we have adopted the setup where the map α acts on densities of normal states, i.e. on positive elements from $L^1(\mathscr{M}, \tau)$. However, a 'dual' situation referring to states instead of their densities is also possible, namely, when for a state ρ the transformed state $\rho \circ \alpha$ is considered. Then for the density of this state we have

$$h_{\rho\circ\alpha} = \widetilde{\alpha}(h_\rho),$$

and it is seen that in order to deal with densities we must consider the map $\widetilde{\alpha}$ instead of α. Taking into account the equality $\widetilde{\widetilde{\alpha}} = \alpha$, the equivalent conditions of Theorem 12 would then read

(i) $H(\rho) = H(\rho \circ \alpha)$,
(ii) $\widetilde{\alpha}$ is a *-isomorphism on the algebra $W^*(h_\rho)$,
(iii) $\alpha(\widetilde{\alpha}(h_\rho)) = h_\rho$.

This approach will be followed in the next section because in measurement theory one traditionally considers maps acting on states as a basic object.

4. Mappings preserving entropy (measurement theory)

A mathematical tool for measurement theory was proposed in 1970 by Davies and Lewis in [3], and is based on the notion of *instrument* which in our context of von Neumann algebras can be briefly described as follows. Let $(\Omega, \mathfrak{F})$ be a measurable space describing the outcomes of a measurement performed on a physical system (usually, $\Omega = \mathbf{R}$, $\mathfrak{F} = \mathcal{B}(\mathbf{R})$), let a von Neumann algebra $\mathscr{M}$ describe the bounded observables of this system, and let $\mathscr{M}_*^+$ be the set of (non-normalised) states of this system. The change of state caused by measurement is described by a map $\mathcal{E}\colon \mathfrak{F} \to \mathcal{L}(\mathscr{M}_*)^+$ which to each set $\Delta \in \mathfrak{F}$ assigns a positive linear map $\mathcal{E}_\Delta$ acting on the predual of $\mathscr{M}$. Then $\mathcal{E}_\Delta\rho$ is the (non-normalised) state of the system, initially in state ρ, after the measurement with the outcome in the set Δ. For the map $\mathcal{E}$, countable additivity and the condition $(\mathcal{E}_\Omega\rho)(\mathbf{1}) = \rho(\mathbf{1})$ for each state $\rho \in \mathscr{M}_*$ are assumed. The map $\mathcal{E}_\Omega$ describes the general change of state under measurement, so that if the system was initially in state ρ, then the state of the system after measurement, without reading its result, is $\mathcal{E}_\Omega\rho$.

Now *repeatable* instruments (considered, without using this name, already by J. von Neumann in his theory of measurement formulated for discrete Ω and the algebra $\mathbf{B}(\mathcal{H})$) are characterised by the condition

$$\mathcal{E}_\Delta^2 = \mathcal{E}_\Delta \quad \text{for each } \Delta \in \mathfrak{F}.$$

Passing to the *dual* instrument $\mathcal{E}^*$, we obtain a map $\mathcal{E}^*\colon \mathfrak{F} \to \mathcal{L}(\mathscr{M})^+$ which to each set $\Delta \in \mathfrak{F}$ assigns a positive normal linear map $\mathcal{E}_\Delta^*$ acting on $\mathscr{M}$. The map $\mathcal{E}^*$ is also countably additive and satisfies the condition $\mathcal{E}_\Omega^*(\mathbf{1}) = \mathbf{1}$. The condition of repeatability has now the form

$$\mathcal{E}_\Delta^{*2} = \mathcal{E}_\Delta^* \quad \text{for each } \Delta \in \mathfrak{F}.$$

(See [2] or [3] for a more detailed description of instruments.)

We have the following characterisation of the states invariant with respect to repeatable measurements in terms of Segal's entropy.

Theorem 13. *Let $\mathcal{E}$ be a repeatable instrument on a von Neumann algebra $\mathscr{M}$ with a normal faithful semifinite trace τ, such that $\tau \circ \mathcal{E}_\Omega^* = \tau$. For an arbitrary $\rho \in \mathscr{M}_*^+$ with finite Segal's entropy, we have*

$$H(\mathcal{E}_\Omega\rho) = H(\rho)$$

if and only if

$$\mathcal{E}_\Omega\rho = \rho.$$

Proof. Let h_ρ be the density of the state ρ, and let $\widetilde{\mathcal{E}_\Omega^*}$ be the map conjugate to $\mathcal{E}_\Omega^*$. In [4, Lemma 2] it was shown that for the density $h_{\mathcal{E}_\Omega\rho}$ of the state $\mathcal{E}_\Omega\rho$ the formula

$$h_{\mathcal{E}_\Omega\rho} = \widetilde{\mathcal{E}_\Omega^*}(h_\rho) \tag{20}$$

holds under the assumption that $h_\rho \in \mathscr{M}$. However, the relation $\tau \circ \widetilde{\mathcal{E}_\Omega^*} = \tau$ yields the possibility to extend the map $\widetilde{\mathcal{E}_\Omega^*}$ to $L^1(\mathscr{M}, \tau)$ obtaining thus the formula (20) for arbitrary $0 \leqslant h_\rho \in L^1(\mathscr{M}, \tau)$.

Assume that

$$H(\mathcal{E}_\Omega\rho) = H(\rho),$$

and let $z = z^* \in W^*(h_\rho)$ be arbitrary. From Remark it follows that $\widetilde{\mathcal{E}^*_\Omega}|W^*(h_\rho)$ is a *-isomorphism, so

$$\widetilde{\mathcal{E}^*_\Omega}(z^* \circ z) = \widetilde{\mathcal{E}^*_\Omega}(z^*) \circ \widetilde{\mathcal{E}^*_\Omega}(z).$$

Proposition 9 yields the equality

$$\widetilde{\mathcal{E}^*_\Omega}(z \circ y) = \widetilde{\mathcal{E}^*_\Omega}(z) \circ \widetilde{\mathcal{E}^*_\Omega}(y)$$

for every $y \in \mathscr{M}$. Further, we have

$$\big(\widetilde{\mathcal{E}^*_\Omega}(z) - z\big)^2 = \widetilde{\mathcal{E}^*_\Omega}(z)^2 - \widetilde{\mathcal{E}^*_\Omega}(z)z - z\widetilde{\mathcal{E}^*_\Omega}(z) + z^2 = \widetilde{\mathcal{E}^*_\Omega}\big(z^2\big) - 2z \circ \widetilde{\mathcal{E}^*_\Omega}(z) + z^2,$$

thus

$$\begin{aligned}\widetilde{\mathcal{E}^*_\Omega}\big((\widetilde{\mathcal{E}^*_\Omega}(z) - z)^2\big) &= \widetilde{\mathcal{E}^*_\Omega}\big(\widetilde{\mathcal{E}^*_\Omega}(z^2)\big) - 2\widetilde{\mathcal{E}^*_\Omega}\big(z \circ \widetilde{\mathcal{E}^*_\Omega}(z)\big) + \widetilde{\mathcal{E}^*_\Omega}\big(z^2\big)\\ &= 2\widetilde{\mathcal{E}^*_\Omega}\big(z^2\big) - 2\widetilde{\mathcal{E}^*_\Omega}(z) \circ \widetilde{\mathcal{E}^*_\Omega}\big(\widetilde{\mathcal{E}^*_\Omega}(z)\big) = 2\widetilde{\mathcal{E}^*_\Omega}\big(z^2\big) - 2\widetilde{\mathcal{E}^*_\Omega}(z)^2 = 0,\end{aligned}$$

and the faithfulness of $\widetilde{\mathcal{E}^*_\Omega}$ yields

$$\widetilde{\mathcal{E}^*_\Omega}(z) = z.$$

Let now $h_n \in W^*(h_\rho)$ be such that $h_n \to h_\rho$ in $\|\cdot\|_1$-norm. We have

$$\widetilde{\mathcal{E}^*_\Omega}(h_n) = h_n,$$

and passing to the limit gives the equality

$$\widetilde{\mathcal{E}^*_\Omega}(h_\rho) = h_\rho,$$

i.e.

$$h_{\mathcal{E}_\Omega \rho} = h_\rho,$$

which finishes the proof. □

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