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SCHOENFLIES SOLUTIONS OF CONFORMAL BOUNDARY VALUES MAY FAIL TO BE SOBOLEV

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Abstract. There exists a planar Jordan domains Ω with 1-Hausdorff dimensional boundary such that, for any conformal map $\varphi\colon \mathbf{D} \to \Omega$, any homeomorphic extensions to the entire plane of either φ or φ^{-1} cannot be in $W^{1,1}_{\text{loc}}$ class (or even not in BV_{loc}).

1. Introduction

Let $\Gamma \subset \mathbf{C}$ be a Jordan curve, namely there exists a homeomorphism $\phi\colon \mathbf{S}^1 \to \Gamma$, where $\mathbf{C}$ is the complex plane and $\mathbf{S}^1$ denotes the boundary of the unit disk $\mathbf{D}$. According to Jordan curve theorem, the curve Γ divides $\mathbf{C}$ into two components, and we call the bounded component a Jordan domain.

Jordan–Schoenflies theorem states that any homeomorphism between two Jordan curves on $\mathbf{C}$ can be extended to a homeomorphism between the entire $\mathbf{C}$; see [12, Corollary 2.9]. To be more precise, given two Jordan domains Ω_1 and Ω_2 and a homeomorphism $\varphi\colon \partial\Omega_1 \to \partial\Omega_2$, there exists a homeomorphism Φ, which we call a *Schoenflies solution* of the boundary value φ, from $\mathbf{C}$ to $\mathbf{C}$ such that the restriction of Φ to Γ_1 coincides with φ. Then a natural question arises:

Question 1.1. Given two Jordan domains $\Omega_1, \Omega_2 \subset \mathbf{C}$ together with a homeomorphism $\varphi\colon \partial\Omega_1 \to \partial\Omega_2$, what is the best regularity of Schoenflies solutions of the boundary value φ?

Certainly the answer to this question depends on the given boundary value and the geometry of both Ω_1 and Ω_2. Let us recall some known results. If Ω_2 is bounded by a smooth Jordan curve, then by the techniques from differential topology for each conformal map we can find a smooth Schoenflies solution to any homeomorphism from $\mathbf{S}^1$ onto $\partial\Omega$. Assume that $\varphi\colon \mathbf{S}^1 \to \partial\Omega_2$ is quasisymmetric, via Douady–Earle extension theorem there exists a K-quasiconformal Schoenflies solution Φ. By [1], we further have that both Φ and Φ^{-1} are in $W^{1,p}_{\text{loc}}(\mathbf{C})$ for any $p < 2K/(K-1)$. Recently Koskela, Pankka and the author have been working on a version of this result for domains satisfying Gehring–Martio conditions [8].

Recall Carathéodory's theorem states that, given any two Jordan domain Ω_1 and Ω_2, every conformal map $\varphi\colon \Omega_1 \to \Omega_2$ can be continuously extended to the boundary as a homeomorphism $\varphi\colon \overline{\Omega}_1 \to \overline{\Omega}_2$. We abuse φ here. In this paper we investigate Question 1.1 with the boundary value given by Carathéodory's theorem, namely a *conformal boundary value*.

The main result of this paper is the following.

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Theorem 1.2. *There exists a Jordan domain $\Omega \subset \hat{\mathbf{C}}$ with 1-Hausdorff dimensional boundary such that, any Schoenflies solution of any conformal boundary value $\varphi\colon \mathbf{S}^1 \to \partial\Omega$ or $\phi\colon \partial\Omega \to \mathbf{S}^1$ is not in $W^{1,1}_{\mathrm{loc}}(\mathbf{C})$ (even not in $BV_{\mathrm{loc}}(\mathbf{C})$).*

This result indicates that, in general, one cannot expect the regularity of Schoenflies solutions to a given boundary value to be better than homeomorphism; even if the boundary value is given by a (extended) conformal map (which is a quite natural choice). Thus, geometric assumptions on the Jordan domain in question and (energy) controls on the boundary value are necessary. One can see e.g. [2, 13, 6, 9] for recent results in this direction. Especially in the very recent paper [10] Koski and Onninen give positive answers to Question 1.1 under certain circumstances.

The notation in the paper is quite standard. The Euclidean distance between two sets A, $B \subset \mathbf{R}^2$ is denoted by $\operatorname{dist}(A, B)$. We denote by $\ell(\gamma)$ the length of a curve γ. For a set $A \subset \mathbf{R}^2$, we write its boundary as ∂A , and its closure as $\overline{A}$, respectively, with respect to the Euclidean topology. We use the notation $\mathcal{H}^1$ for 1-dimensional Hausdorff measure.

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2. Proof of Theorem 1.2

Define the *inner distance with respect to* Ω between x, $y \in \Omega$ by

$$\operatorname{dist}_{\Omega}(x, y) = \inf_{\gamma \subset \Omega} \ell(\gamma),$$

where the infimum runs over all curves joining x and y in Ω.

2.1. Schoenflies solution of conformal boundary value $\varphi\colon \mathbf{S}^1 \to \partial\Omega$. The idea of the proof is that, we construct a Jordan domain $\Omega \subset \hat{\mathbf{C}}$ satisfying that there exists a (Cantor) set $E \subset \partial\Omega$ such that,

(i) for any conformal $\varphi\colon \mathbf{D} \to \Omega$, i.e. for any conformal boundary value, we have

$$\mathcal{H}^1(\varphi^{-1}(E)) > 0;$$

(ii) for any point x in the complementary domain $\widetilde{\Omega}$,

$$\operatorname{dist}_{\widetilde{\Omega}}(x, E \setminus \{(0, 0), (1, 0)\}) = \infty.$$

If such a Jordan domain exists (see Lemma 2.1 below), then by (i) and (ii), any Schoenflies solution of the conformal boundary value φ is not in $W^{1,1}_{\mathrm{loc}}$ (even not in BV_{loc}) by Fubini's theorem; indeed, such a solution maps a family of radial segments in the exterior of the unit disk (with finite length) into a family curves of infinite length in $\widetilde{\Omega}$. By calculating in the polar coordinate we know that such a map cannot be in $W^{1,1}_{\mathrm{loc}}$ (even not in BV_{loc}). Hence Theorem 1.2 follows.

We first construct a Jordan curve Γ in the plane. Towards this, let us recall the construction of a fat Cantor set $E \subset [0, 1]$ on the real axis. Let $C_0 = I_{0,1} = [0, 1]$ and C_i with $i \geq 1$ recursively as follows: When $I_{i,j} = [a, b]$ has been defined, let

$$I_{i+1, 2j-1} = \left[a, \frac{a+b-4^{-i}}{2}\right] \quad \text{and} \quad I_{i+1, 2j} = \left[\frac{a+b+4^{-i}}{2}, b\right];$$

i.e. we remove an open interval of length 4^{-i} from the middle of the interval $I_{i,j}$. Then we set

$$C_i = \bigcup_{j=1}^{2^i} I_{i,j}.$$

The set E is finally given by

$$E = \bigcap_{i=1}^{\infty} C_i.$$

A simple calculation shows that, for every $i \in \mathbf{N}$ and $1 \le j \le 2^i$, each interval $I_{i,j}$ has length

$$\frac{2^i+1}{2^{2i+1}} \in (2^{-i-1}, 2^{-i}]. \tag{2.1}$$

Thus C_i, and hence E is well-defined. Moreover, E has positive $\mathcal{H}^1$-measure; note that at step i, $i \ge 1$ there are 2^i intervals removed with total length 2^{-i-1}.

We now construct a sequence of simple curves γ_i based on the construction of E. Again we proceed inductively according to the index i. For $i \in \mathbf{N}$ and $1 \le j \le 2^i$, denote by $I'_{i,j} \subset I_{i,j}$ the interval removed from $I_{i,j}$ in the construction of E. Let γ_0 be the interval $[0, 1]$. When γ_{i-1}, $i \ge 1$ has been defined, we replace every open interval $I'_{i,j}$, $1 \le j \le 2^i$, contained in γ_{i-1}, by a curve

$$\gamma_{i,j} = \partial(I'_{i,j} \times [0, 2^{-i}]) \setminus (I'_{i,j} \times \{0\}),$$

consisting of three line segments, where $\times$ means the Cartesian product. We then obtain γ_i. See Figure 1.

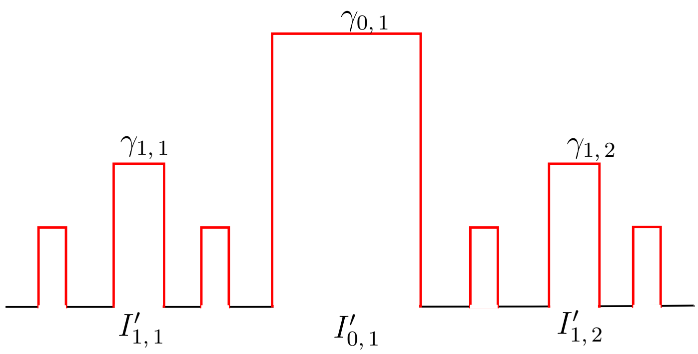

Figure 1. The curve γ_2 is shown in the figure. In the previous steps the intervals $I'_{0,1}$, $I'_{1,1}$, $I'_{1,2}$ were replaced by curves $\gamma_{0,1}$, $\gamma_{1,1}$, $\gamma_{1,2}$, and in the current step four more intervals are replaced.

Since $\{\gamma_i\}$ (under suitable parameterizations) is a Cauchy sequence of curves in the plane with respect to the supremum distance, the limit γ exits and is a curve. Moreover, γ is simple.

For fixed $n \in \mathbf{N}$, there are $2^{n+1} - 1$ curves $\gamma_{i,j}$ intersecting $\mathbf{R} \times (2^{-n-1}, 2^{-n}]$. Indeed, if $\gamma_{i,j} \cap (\mathbf{R} \times (2^{-n-1}, 2^{-n}]) \ne \emptyset$, then $i \le n$. The distance between any two of these curves is strictly larger than 2^{-n-1} by (2.1).

We next construct a sequence of new curves Γ_n according to the index n. First of all define $\Gamma_0 = \gamma$. When Γ_{n-1}, $n \ge 1$ has been defined, we modify the segments in

$$\gamma_{i,j} \cap (\mathbf{R} \times (2^{-n-1}, 2^{-n}]), \quad 0 \le i \le n,$$

to obtain Γ_n. Recall that $\gamma_{i,j}$ replaces the interval $I'_{i,j}$ in the construction of γ_i. Denote by $a_{i,j}$ and $b_{i,j}$ the end points of $I'_{i,j}$ with $a_{i,j} < b_{i,j}$. Then for every $1 \le i \le n$,

$$\gamma_{i,j} \cap (\mathbf{R} \times (2^{-n-1}, 2^{-n}]) = (\{a_{i,j}\} \times (2^{-n-1}, 2^{-n}]) \cup (\{b_{i,j}\} \times (2^{-n-1}, 2^{-n}])$$

and each $1 \le k \le 2^n - 1$, we replace each segment

$$\{a_{i,j}\} \times [2^{-n-1} + (4k)2^{-2n-3},\, 2^{-n-1} + (4k+1)2^{-2n-3}]$$

by

$$A^{n,k}_{i,j} := \partial\left([a_{i,j} - 2^{-n-1},\, a_{i,j}] \times [2^{-n-1} + (4k)2^{-2n-3},\, 2^{-n-1} + (4k+1)2^{-2n-3}]\right) \setminus \{a_{i,j}\} \times [2^{-n-1} + (4k)2^{-2n-3},\, 2^{-n-1} + (4k+1)2^{-2n-3}],$$

and

$$\{b_{i,j}\} \times [2^{-n-1} + (4k+2)2^{-2n-3},\, 2^{-n-1} + (4k+3)2^{-2n-3}]$$

by

$$B^{n,k}_{i,j} := \partial\left([b_{i,j},\, b_{i,j} + 2^{-n-1}] \times [2^{-n-1} + (4k+2)2^{-2n-3},\, 2^{-n-1} + (4k+3)2^{-2n-3}]\right) \setminus \{b_{i,j}\} \times [2^{-n-1} + (4k+2)2^{-2n-3},\, 2^{-n-1} + (4k+3)2^{-2n-3}].$$

This gives us the new curve Γ_n. See Figure 2.

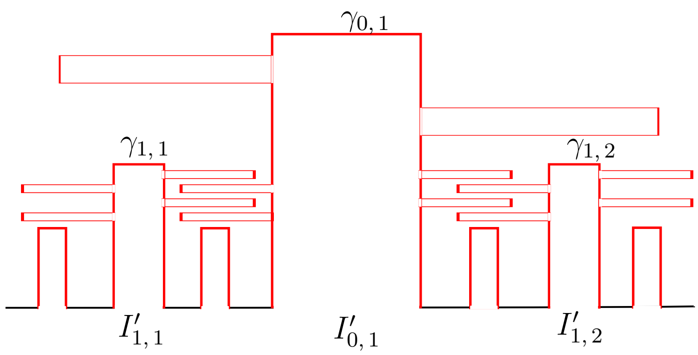

Figure 2. The curve Γ_2 is shown in the figure, with the replacement of certain segments contained in Γ_1 by parts of boundaries of some rectangles, receptively.

Again since $\{\Gamma_n\}$ (under suitable parameterizations) is a Cauchy sequence of curves in the plane with respect to the supremum distance, we conclude that Γ_n converges uniformly to some curve Γ_∞ as $n \to \infty$. Moreover, according to our construction Γ_∞ is simple. Define

$$\Gamma = \Gamma_\infty \cup (\partial([0,\, 1] \times [-1,\, 0]) \setminus [0,\, 1] \times \{0\}).$$

Since Γ_∞ is simple, then also Γ is simple, and hence Jordan as it is closed. We denote by Ω the bounded component of $\mathbf{C} \setminus \Gamma$.

Notice that $\partial\Omega$ is a countable union of rectifiable curves, even though it does not have finite length. Since the Hausdorff dimension of a countable union of sets is the supremum of the Hausdorff dimensions of the sets, see e.g. [11, Page 81, Section 5.9], we conclude that $\partial\Omega$ is a set of Hausdorff dimension 1.

Recall the Cantor set E. Now let us check that the Jordan domain Ω satisfies the two properties (i) and (ii). We remark that, in our construction, for any point x in Ω, we have

$$\operatorname{dist}_\Omega(x,\, E) < \infty.$$

Before showing (i), we note that, property (i) is stated with respect to all conformal maps. However, since two such Riemann maps differ from each other by a Möbius transform on the unit disk, we may assume that $\varphi(0)$ is the center of the square $[0,\, 1] \times [-1,\, 0]$.

Recall that the harmonic measure in the unit disk is defined via the Poisson kernel, and then in any Jordan domain via the (extended) Riemann mapping. For

$E \subset \partial\Omega$, we use $\omega(x_0, E, \Omega)$ to designate *the harmonic measure of* E *at* x_0 *in* Ω. It is known that $\omega(x_0, E, \Omega) = u(x_0)$ where u is the (unique) harmonic function in Ω whose boundary value is the characteristic function of E on $\partial\Omega$. We refer to [4] for more details.

Lemma 2.1. *The Jordan domain* Ω *constructed above satisfies properties (i) and (ii).*

Proof. Towards (i), we first observe that

$$\omega(\varphi(0), E, \Omega) \geq \omega(\varphi(0), E, Q) > 0, \tag{2.2}$$

where Q is the open square $(0, 1) \times (-1, 0)$. Indeed, the first estimate comes from the comparison principle of harmonic measures, while the second inequality follows from F. and M. Riesz theorem since its 1-Hausdorff measure is strickly positive.

By the conformal invariance of harmonic measure we have

$$\omega(0, \varphi^{-1}(E), \mathbf{D}) > 0.$$

According to the definition of harmonic measures in the unit disk, we conclude (i).

To show (ii), note that in our construction, any curve in the unbounded component of $\mathbf{R}^2 \setminus \Gamma$ towards $E \setminus \{(0, 0), (1, 0)\}$ has length at least $\frac{1}{2}$ in the region $\mathbf{R} \times (2^{-n-1}, 2^{-n}]$ for n large enough; the curve has to oscillate 2^n times and each time it goes at least 2^{-n-1}. This implies that any curve in the unbounded component of $\mathbf{R}^2 \setminus \Gamma$ towards $E \setminus \{(0, 0), (1, 0)\}$ has infinite length. Property (ii) is complete. ☐

2.2. Schoenflies solution of conformal boundary value $\phi\colon \partial\Omega \to \mathbf{S}^1$. Let $\phi\colon \Omega \to \mathbf{D}$ be a conformal map giving the conformal boundary value via Carathéodory's theorem. By composing with a suitable Möbius map, we may assume that $\phi(z_0) = 0$, where z_0 is the center of open square $Q = (0, 1) \times (-1, 0)$; in the general case the constants below will further depending on the Möbius transform. We show that any homeomorphic extension of ϕ is not in $W^{1,1}_{\text{loc}}$.

Towards this, recall that in the construction of $\Gamma = \partial\Omega$ we attached "arms" $A^{n,k}_{i,j}$ and $B^{n,k}_{i,j}$ to every curve $\gamma_{i,j}$. We first claim that, there exists an absolute constant $c > 0$ such that, for $n \geq 3$,

$$\operatorname{dist}(\phi(A^{n,k}_{i,j}), \phi(B^{n,k'}_{i',j'})) \geq c2^{-n} \tag{2.3}$$

whenever $I'_{i,j}, I'_{i',j'} \subset \left[\frac{5}{32}, \frac{27}{32}\right]$ and either $i \neq i'$ or $j \neq j'$.

Indeed, let us fix $A^{n,k}_{i,j}$ and $B^{n,k'}_{i',j'}$. According to our construction, there exists an interval $J \in \{I_{n+1,j}\}_{j=1}^{2^{n+1}}$ such that $J \subset \left[\frac{5}{32}, \frac{27}{32}\right]$ is between $I'_{i,j}$ and $I'_{i',j'}$. Since $\phi\colon \partial\Omega \to \mathbf{S}^1$ is a homeomorphism, by the construction of $\partial\Omega$ and the geometry of the unit circle, we have that

$$\operatorname{dist}(\phi(A^{n,k}_{i,j}), \phi(B^{n,k'}_{i',j'})) \geq c_1\mathcal{H}^1(\phi(J \cap E))$$

for some absolute constant c_1. Therefore it suffices to show that $\mathcal{H}^1(\phi(J \cap E)) \geq c_2 2^{-n}$ for some absolute constant c_2.

Again by the invariance of harmonic measure under conformal map and the comparison principle of harmonic measures,

$$\omega(0, \phi(J \cap E), \mathbf{D}) = \omega(z_0, J \cap E, \Omega) \geq \omega(z_0, J \cap E, Q).$$

According to Schwarz–Christoffel formula [12, Chapter 3.1], since $J \subset \left[\frac{5}{32}, \frac{27}{32}\right]$ is away from the corner of Q, we have

$$\omega(z_0, J, Q) \geq c_3 2^{-n}$$

for some absolute constant c_3; note that the length of J is $2^{-n-2} + 2^{-1}4^{-n-1}$, and E is a self-similar fat Cantor set. Therefore, we conclude (2.3) via the Poisson formula in the unit disk.

Let Φ be any Schoenflies solution of the conformal boundary value ϕ. By (2.3), the image under ϕ of any vertical segment joining "neighboring arms" $A^{n,k}_{i,j}$ and $B^{n,k}_{i',j'}$ in the exterior of Ω has length at least $c2^{-n}$. Moreover, when $n \geq 3$, the intersection of the projections on the real axis of the "neighboring arms" $A^{n,k}_{i,j}$ and $B^{n,k}_{i',j'}$ is an interval with length not less than 2^{-n-2}, and there are at least 4^n pairs of those "neighboring arms" contained in $\left[\frac{5}{32}, \frac{27}{32}\right] \times [2^{-n-1}, 2^{-n}]$ up to a multiplicative constant. Therefore by Fubini's theorem we conclude

$$\int_{\left[\frac{5}{32}, \frac{27}{32}\right] \times [2^{-n-1}, 2^{-n}]} |D\Phi| \, dx \geq c' 2^{-n} 2^{-n-2} 4^n \geq c' 2^{-2}$$

for some absolute constant c'. Therefore, in any Euclidean neighborhood of $E \cap \left[\frac{5}{32}, \frac{27}{32}\right]$ the $W^{1,1}_{\mathrm{loc}}$-energy of Φ is infinite, and a similar argument shows that $\Phi \notin BV_{\mathrm{loc}}$. This concludes the second part of Theorem 1.2.

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