

POINCARÉ-TYPE INEQUALITIES IN MUSIELAK SPACES

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Abstract. In this paper we investigate Poincaré-type integral inequalities in the functional Musielak structure. We extend the ones already well known in Sobolev, Orlicz and variable exponent Sobolev spaces. We introduce conditions on the Musielak functions under which they hold. The identification with null trace functions space is given.

1. Introduction and main results

In the last two decades, there has been an increasingly interest in studying Musielak spaces, particularly for the analysis of nonlinear partial differential equations with non-standard growth conditions which come from modelling modern materials such as non Newtonian fluids, see for instance [9, 19] and the references therein.

In [10, 11, 17] there is a basic background on the Musielak spaces $L_M(\Omega)$ and the Musielak–Sobolev spaces $W^m L_M(\Omega)$. An interesting missing feature is the Poincaré-type inequalities (in norm or in integral forms) in the closed subspace $W_0^m L_M(\Omega)$ defined as the closure of the set $C_0^\infty(\Omega)$ of compactly supported functions in Ω with respect to the weak-* topology $\sigma(\Pi L_M, \Pi E_{M^*})$ in the Musielak–Orlicz space $W^m L_M(\Omega)$. However, proving the Poincaré integral inequality for functions in $C_0^\infty(\Omega)$ and then extending it by a density argument (as is often done for a constant exponent) is not an easy task since the passage to the limits is not allowed because of the lack in general of density of smooth functions in $W_0^m L_M(\Omega)$ at least in the modular sense (see Definition 2.1). This is mainly due to the fact that the shift operator is not acting in general on Musielak spaces unless some regularity conditions on the Musielak function M are satisfied see [2, 21].

In this paper, we are interested in the problem of Poincaré-type integral inequality in the Musielak spaces. Such integral inequality yields obviously the Poincaré norm inequality. Precisely, we give sufficient conditions on the Φ -function M for the following Poincaré-type inequality

$$\int_{\Omega} \sum_{|\alpha| < m} M(x, |D^\alpha u(x)|) dx \leq \int_{\Omega} \sum_{|\alpha| = m} M(x, c|D^\alpha u(x)|) dx$$

to hold for every $u \in W_0^m L_M(\Omega)$ where $c > 0$ is a constant. We also get the same inequality in the subspace $W_0^m E_M(\Omega)$ under minimal assumptions.

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1.1. Poincaré-type inequalities: state of the art. Let Ω be a bounded open subset of \mathbf{R}^N , $N \geq 1$ and let $1 \leq p < \infty$. The usual Sobolev spaces are denoted $W^{1,p}(\Omega)$ while by $W_0^{1,p}(\Omega)$ we denote the norm closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The classical Poincaré integral inequality asserts that

$$(1.1) \quad \int_{\Omega} |u(x)|^p dx \leq C(\Omega, p) \int_{\Omega} |\nabla u(x)|^p dx$$

for every $u \in W_0^{1,p}(\Omega)$ where $C(\Omega, p)$ is a constant depending on Ω and p . In fact, this inequality remains valid if Ω is only bounded in one direction. Recalling here that when Ω is regular (see for instance [18, Theorem 4.14]) we have

$$(1.2) \quad W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \text{tr}(u) = 0 \text{ on } \partial\Omega\}$$

and hence $W_0^{1,p}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,p}(\Omega)$.

Gossez [8, Lemma 5.7] proved the existence of two constants $c_m > 0$ and $c_{m,\Omega} > 0$ such the following Orlicz version of Poincaré integral inequality

$$(1.3) \quad \int_{\Omega} \sum_{|\alpha| < m} \varphi(|D^\alpha u(x)|) dx \leq c_m \int_{\Omega} \sum_{|\alpha|=m} \varphi(c_{m,\Omega}|D^\alpha u(x)|) dx,$$

holds for every $u \in W_0^m L_\varphi(\Omega)$. Here, $W_0^m L_\varphi(\Omega)$ is defined as the closure of the set $C_0^\infty(\Omega)$ of compactly supported functions in Ω with respect to the weak-* topology $\sigma(\Pi L_\varphi, \Pi E_{\varphi^*})$ in the Orlicz spaces $W^m L_\varphi(\Omega)$, where φ and φ^* form a pair of complementary N -functions, cf. [1]. Since no extra condition is assumed on φ , inequality (1.3) proved in $W_0^m L_\varphi(\Omega)$ covers not only (1.1) but it remains valid for a wide class of Orlicz functions. In contrast to Sobolev spaces $W_0^{1,p}(\Omega)$, the introduction of the Orlicz spaces $W_0^m L_\varphi(\Omega)$, defined by mean of the weak-* topology $\sigma(\Pi L_\varphi, \Pi E_{\varphi^*})$, seems to be more convenient and very interesting in the theory of existence of PDEs in nonreflexive functional spaces, since firstly the weak topology is not equivalent in general to the strong one and secondly coarser topology has more compact sets than the strong one.

Unfortunately, in the framework of variable exponent spaces the situation is more complicated and more regularities on the exponent are needed. In fact in the Sobolev space $W_0^{1,p(\cdot)}(\Omega)$, defined as the norm closure of $C_0^\infty(\Omega)$ functions in $W^{1,p(\cdot)}(\Omega)$, the Poincaré norm inequality was first proved in the pioneering paper [15, Theorem 3.10] written about variable exponent Sobolev spaces provided that the exponent $p(\cdot)$ is continuous on $\overline{\Omega}$ and then by using the approche based on the boundedness of the maximal operator on $L^{p(\cdot)}(\Omega)$, the authors in [4, Theorem 6.21] proved the Poincaré norm inequality

$$(1.4) \quad \|u\|_{L^{p(\cdot)}(\Omega)} \leq c(N, p(\cdot), \Omega) \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for every $u \in W_0^{1,p(\cdot)}(\Omega)$ and for exponents $p(\cdot)$ satisfying $1 < p^- \leq p(x) \leq p^+ < +\infty$ and the so-called log-Hölder regularity, that is

$$(1.5) \quad |p(x) - p(y)| \leq \frac{-C_0}{\log(|x - y|)}; \quad \text{for every } x, y \in \Omega \text{ with } |x - y| \leq \frac{1}{2},$$

for some constant $C_0 > 0$.

In [5, Theorem 8.2.4] the authors defined $W_0^{1,p(\cdot)}(\Omega)$ as the closure of Sobolev functions with compact support in Ω with respect to the norm in $W^{1,p(\cdot)}(\Omega)$. They

proved the Poincaré norm inequality for a regular bounded domain and for exponent $p(\cdot)$ satisfying the following two conditions

$$(1.6) \quad \left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{C_1}{\log(e + \frac{1}{|x-y|})}$$

and for some $p_\infty \in \mathbf{R}$

$$(1.7) \quad \left| \frac{1}{p(x)} - p_\infty \right| \leq \frac{C_2}{\log(e + |x|)}$$

for every $x, y \in \Omega$, where $C_1 > 0$ and $C_2 > 0$ are constants. Let us note in passing that the above two definitions of the Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ coincide if $p(\cdot)$ is a measurable bounded exponent and if (1.6) and (1.7) are fulfilled (see [5, Corollary 11.2.4]). Ciarlet and Dinca [3] proved the Poincaré norm inequality using an approach which does not rely on the density arguments.

In general in the variable exponent Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$ (defined as the norm closure of $C_0^\infty(\Omega)$ functions in $W^{1,p(\cdot)}(\Omega)$), the Poincaré integral inequality (1.1) with variable exponent $p(\cdot)$ instead of constant exponent p fails to hold as it was shown in [6, Example, pp. 444–445]. Indeed, if the variable exponent $p(\cdot)$ is a continuous function having a minimum or a maximum then an integral version of the Poincaré inequality can not be obtained (see [20]). However, under a suitable monotony property on the variable exponent $p(\cdot)$ Maeda [16] proved the Poincaré integral inequality for $C_0^1(\Omega)$ -functions.

It is worth recalling that the Poincaré norm inequality in $W_0^{1,p(\cdot)}(\Omega)$ obtained in the aforementioned references requires the continuity of the variable exponent. Here we prove the Poincaré integral inequality in Musielak spaces, and so the Poincaré norm inequality, by introducing some assumptions that don't require the continuity of the variable exponent when reducing to Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$.

1.2. Structural assumptions. In this subsection we give the definition of Musielak Φ -functions and we introduce new systematic sufficient conditions which enable us to prove Poincaré-type integral inequalities in Musielak spaces.

Definition 1.1. (ϕ -function, Φ -function). A real function $M: \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is called a ϕ -function, written $M \in \phi$, if $M(x, \cdot)$ is a nondecreasing and convex function for all $x \in \Omega$ with $M(x, 0) = 0$, $M(x, s) > 0$ for $s > 0$, $M(x, s) \rightarrow \infty$ as $s \rightarrow \infty$ and $M(\cdot, s)$ is a measurable function for every $s \geq 0$.

A ϕ -function is called Φ -function, written $M \in \Phi$, if furthermore it satisfies $\text{ess inf}_{x \in \Omega} M(x, 1) > 0$ and

$$\lim_{s \rightarrow 0} \frac{M(x, s)}{s} = 0 \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{M(x, s)}{s} = \infty.$$

Throughout the paper, we consider Φ -functions on which we assume at least one of the following fundamental regularity assumptions.

(M1) There exists a function $\varphi: [0, 1/2] \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $\varphi(\cdot, s)$ and $\varphi(x, \cdot)$ are nondecreasing functions and for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \frac{1}{2}$ and for any constant $c > 0$

$$M(x, s) \leq \varphi(|x - y|, s)M(y, s), \quad \text{with} \quad \limsup_{\varepsilon \rightarrow 0^+} \varphi(\varepsilon, c\varepsilon^{-N}) < \infty.$$

(M2) A Φ -function M is said to satisfy the Y -condition on a segment $[a, b]$ of the real line \mathbf{R} , if

either

$$(Y_0) : \begin{cases} \text{there exist } t_0 \in \mathbf{R}^+ \text{ and } 1 \leq i \leq N \text{ such that the partial function} \\ x_i \in [a, b] \mapsto M(x, t) \text{ changes constantly its monotony on both} \\ \text{sides of } t_0 \text{ (that is for } t \geq t_0 \text{ and } t < t_0), \end{cases}$$

or

$$(Y_\infty) : \begin{cases} \text{there exists } 1 \leq i \leq N \text{ such that for all } t \geq 0, \text{ the partial function} \\ x_i \in [a, b] \mapsto M(x, t) \text{ is monotone on } [a, b]. \end{cases}$$

Here, x_i stands for the i^{th} component of $x \in \Omega$.

Example 1.1. A Φ -function $M: \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ satisfies the (Y_0) -condition on $[a, b] \subset \Omega$ with respect to x_1 if the following hold:

- (i) $O = \{x' = (x_2, \dots, x_N) \in \mathbf{R}^{N-1}; (x_1, x') \in \Omega \text{ for any } x_1 \in [a, b]\}$ is non-empty. (Consequently for $t \in \mathbf{R}^+$, the partial mapping $x_1 \in [a, b] \rightarrow M((x_1, x'), t) \in \mathbf{R}^+$ makes sense for any $x' \in O$).
- (ii) There exists $t_0 > 0$ such that for any $x' \in O$, if $t \geq t_0$ the map $x_1 \in [a, b] \rightarrow M((x_1, x'), t) \in \mathbf{R}^+$ has certain type of monotony, for example it is increasing (decreasing), while for $t < t_0$ the map $x_1 \in [a, b] \rightarrow M((x_1, x'), t)$ decreasing (increasing).

More particular, assume that $p: \Omega \rightarrow \mathbf{R}^+$ satisfies $1 < p(\cdot) < \infty$ and the following:

- (iii) $O = \{x' = (x_2, \dots, x_N) \in \mathbf{R}^{N-1}; (x_1, x') \in \Omega \text{ for any } x_1 \in [a, b]\}$ is non-empty.
- (iv) For any $x' \in O$ the mapping $x_1 \in [a, b] \rightarrow p(x_1, x')$ is non-decreasing.

Take $M(x, t) = t^{p(x)}$, $x \in \Omega$, $t \geq 0$ and let $t_0 = 1$. Then, for $t \geq 1$ $x_1 \in [a, b] \rightarrow M(x, t) = t^{p(x_1, x')}$ is non-decreasing for any $x \in \Omega$ while for $t < 1$, $x_1 \in [a, b] \rightarrow M(x, t) = t^{p(x_1, x')}$ is non-increasing for any $x \in \Omega$. Consequently, $M(x, t) = t^{p(x)}$ satisfies the (Y_0) -condition on $[a, b]$ with $i = 1$.

The highly challenging and important part of the analysis in Musielak spaces is giving a relevant structural condition yielding approximation properties of these nonstandard spaces. In general for a Φ -function M , smooth functions are not dense in norm in the Musielak space $W^m L_M(\Omega)$. The authors [2] introduced the condition $(\mathcal{M}1)$ to study the problem of density of smooth functions in Musielak spaces and they showed that this condition unify and improve the known results in Orlicz–Sobolev spaces as well as the variable exponent Sobolev spaces. In fact, the condition $(\mathcal{M}1)$ holds trivially in the case of Orlicz spaces while in the case of variable exponent Sobolev spaces $(\mathcal{M}1)$ holds if we choose

$$\varphi(\tau, s) = \max \{s^{\sigma(\tau)}, s^{-\sigma(\tau)}\}.$$

When $\sigma(\tau) = -c/\log \tau$, with $0 < \tau \leq 1/2$, we obtain the log-Hölder continuity condition (1.5). Nonetheless, we can choose various φ s. For more examples of Φ -functions satisfying $(\mathcal{M}1)$ we refer to [2].

Remark 1.1. (1) In the case where $M(x, t) = t^{p(x)}$, the assumption (Y_0) prevents the variable exponent $p(\cdot)$ to get a local extremum while (Y_∞) is not satisfied unless $p(\cdot)$ is a constant function.

(2) Let us consider the double phase function $M(x, t) = t^p + a(x)t^q$. If there is $1 \leq i \leq N$ such that the function $x_i \mapsto a(x)$ is monotone then M satisfies obviously

(Y_∞) and so (M2). If $x_i \mapsto a(x)$ is not a constant function then the double phase function M can not satisfy (Y_0).

(3) If $1 < p(\cdot) < +\infty$ and there exists $1 \leq i \leq N$ such that the function $x_i \mapsto p(x)$ is monotone on a compact subset of the real line \mathbf{R} , then the following Φ -functions

$$M_1(x, t) = t^{p(x)}, \quad M_2(x, t) = t^{p(x)} \log(e + t), \quad M_3(x, t) = e^{t^{p(x)}} - 1,$$

satisfy (M2).

We note in passing here, that the assumption (M2) covers the one given in [16]. In what follows, we will need the following local integrability condition on M , that is for any constant number $c > 0$ and for every compact set $K \subset \Omega$

$$(1.8) \quad \int_K M(x, c) \, dx < \infty.$$

Inequality (1.8) was introduced in [17, Definition 7.5] for measurable subsets of Ω with finite measure. Observe that (1.8) is not always satisfied as shown by the following example. Set $\Omega = (-1/2, 1/2)$ and set

$$M(x, s) = \begin{cases} s^{1/x}, & x \in (0, 1/2), \\ s^2, & x \in (-1/2, 0). \end{cases}$$

Note that M is a Φ -function. Consider the compact set $K = [0, 1/4]$, which is contained in Ω . Then for $c > 1$

$$\int_K M(x, c) \, dx = \int_0^{1/4} c^{1/x} \, dx = +\infty.$$

We point out that the assumption (M1) implies (1.8).

1.3. Main results. In this subsection we give our main results. Let M be a Φ -function and let M^* be its complementary Φ -function (see (2.1) hereafter). Assume that M and M^* satisfy both the condition (1.8). Then by [21] we can define the space $W_0^m L_M(\Omega)$ to be the $\sigma(\Pi L_M, \Pi E_{M^*})$ closure of $C_0^\infty(\Omega)$ in $W^m L_M(\Omega)$.

The first result we obtain concerns Poincaré-type inequalities in the Musielak spaces $W_0^m L_M(\Omega)$.

Theorem 1.1. *Let Ω be a bounded open subset in \mathbf{R}^N having the segment property. Let M and M^* be a pair of complementary Φ -functions such that M satisfies (M1) and (M2) and M^* satisfies (1.8). Then there exists a constant $c_{m,\Omega}$ depending only on m and Ω such that for every $u \in W_0^m L_M(\Omega)$*

$$(1.9) \quad \int_\Omega \sum_{|\alpha| < m} M(x, |D^\alpha u|) \, dx \leq \int_\Omega \sum_{|\alpha| = m} M(x, c_{m,\Omega} |D^\alpha u|) \, dx.$$

Moreover, for every $u \in W_0^m L_M(\Omega)$

$$(1.10) \quad \sum_{|\alpha| < m} \|D^\alpha u\|_{M,\Omega} \leq C(m, \Omega) \sum_{|\alpha| = m} \|D^\alpha u\|_{M,\Omega},$$

where $C(m, \Omega)$ is a constant depending only on m and Ω .

In the framework of Orlicz spaces, Theorem 1.1 was proved by Gossez [8, Lemma 5.7] where only the definition of the space $W_0^m L_\varphi$, defined as the closure of $C_0^\infty(\Omega)$ -functions with respect to the weak-* topology $\sigma(\Pi L_\varphi, \Pi E_{\varphi^*})$, is used to get the Poincaré integral inequality without assuming the segment property on the bounded open Ω . As in the classical way, the Poincaré integral inequality was first proved for

smooth functions and then (1.9) follows by a density argument based on mollifications. In general, the shift operator is not acting between Musielak spaces (see [15, example 2.9 and Theorem 2.10]). So we face a major difficulty in using mollification and then we can not use the same approach as in [8].

Our contribution to overcome this problem consists in using the regularity condition $(\mathcal{M}1)$ on the Φ -function M and the segment property on the domain Ω (see Definition 2.2). Those conditions enable us to get the modular density of C_0^∞ -functions in $W_0^m L_M(\Omega)$ (see [2]) which we use in the proof of Theorem 1.1 instead the weak- $*$ density as it was done in [8].

Remark 1.2. A direct consequence of the inequality (1.10), is that the maps

$$u \in W_0^m L_M(\Omega) \rightarrow \sum_{|\alpha| \leq m} \|D^\alpha u\|_{M,\Omega}$$

and

$$u \in W_0^m L_M(\Omega) \rightarrow \sum_{|\alpha|=m} \|D^\alpha u\|_{M,\Omega}$$

define equivalent norms on $W_0^m L_M(\Omega)$.

The following theorem concerns the Poincaré integral inequality in the Musielak–Sobolev space $W_0^m E_M(\Omega)$ defined as the norm closure of C_0^∞ -functions in $W^m E_M(\Omega)$.

Theorem 1.2. *Let Ω be a bounded open subset in \mathbf{R}^N and let $M \in \Phi$ satisfy (1.8) and $(\mathcal{M}2)$. The inequality (1.9) holds true for every $u \in W_0^m E_M(\Omega)$ and then so is (1.10).*

We note here that by the definition of $W_0^m E_M(\Omega)$, we do not need to assume in the above Theorem 1.2 the segment property on Ω and the condition $(\mathcal{M}1)$ on the Φ -function M . Therefore, in view of Remark 1.1 the result we obtain covers the Poincaré integral inequality obtained by Maeda [16] for \mathcal{C}_0^1 -functions in the case where the variable exponent $p(\cdot)$ is assumed to satisfy a monotony condition.

Let $K_0^m L_M(\Omega)$ be the norm closure of the set of $W^m L_M(\Omega)$ functions with compact support in Ω . In the particular case $M(x, t) = t^{p(x)}$, $K_0^m L_M(\Omega)$ is nothing but the space $W_0^{m,p(x)}(\Omega)$ defined in [5].

Theorem 1.3. *Let Ω be an open subset in \mathbf{R}^N and let $M \in \Phi$ satisfy $(\mathcal{M}1)$. Then $K_0^m L_M(\Omega)$ coincides with $W_0^m E_M(\Omega)$. Furthermore, if Ω is bounded and M satisfies $(\mathcal{M}2)$, then (1.9) and (1.10) are fulfilled.*

Now, the remaining question is how to provide a satisfactory generalization of the Poincaré inequality for constant exponent, because the equality (1.2) is substituted by the inclusion

$$W_0^1 L_M(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{\sigma(\Pi L_M, \Pi E_{M^*})} \subset \{u \in W^1 L_M(\Omega) : \text{tr}(u) = 0 \text{ on } \partial\Omega\}$$

which may be strict in general unless additional conditions are imposed on the Φ -function M . Note here that for every $u \in W^1 L_M(\Omega)$ the trace $\text{tr}(u) = u|_{\partial\Omega}$ is well defined. Indeed, if Ω is of finite Lebesgue measure one has $W^1 L_M(\Omega) \hookrightarrow W^{1,1}(\Omega)$ and by the Gagliardo trace theorem (see [7]) we have the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$. Hence, we conclude that for all $u \in W^1 L_M(\Omega)$ there holds $u|_{\partial\Omega} \in L^1(\partial\Omega)$. We give the answer in the following theorem.

Theorem 1.4. *Let Ω be a bounded open subset in \mathbf{R}^N having the segment property. Assume that the pair of complementary Φ -functions M, M^* is such that*

M satisfies (M1) and M^* satisfies (1.8). Then, we get

$$W_0^m L_M(\Omega) = W_0^{m,1}(\Omega) \cap W^m L_M(\Omega).$$

If furthermore Ω has a Lipschitz boundary $\partial\Omega$, then we obtain

$$(1.11) \quad W_0^1 L_M(\Omega) = \{u \in W^1 L_M(\Omega) : \text{tr}(u) = 0 \text{ on } \partial\Omega\}.$$

1.4. Organization of the paper. In section 2 we review some basic facts we use about Musielak spaces. Further details can be found in the standard monograph by Musielak [17] and the papers by Kamińska [12, 13, 14]. Section 3 is devoted to the proof of the main results.

2. Musielak structure

In the following section we give a brief basic review on Musielak–Orlicz spaces. For $M \in \phi$, the Musielak–Orlicz space $L_M(\Omega)$ (resp. $E_M(\Omega)$) is defined as the set of all measurable functions $u: \Omega \rightarrow \mathbf{R}$ such that $\int_{\Omega} M(x, |u(x)|/\lambda) dx < +\infty$ for some $\lambda > 0$ (resp. for all $\lambda > 0$). Equipped with the Luxemburg norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

$L_M(\Omega)$ is a Banach space [17, Theorem 7.7] and $E_M(\Omega)$ is its closed subset. Define $M^*: \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ by

$$(2.1) \quad M^*(x, s) = \sup_{t \geq 0} \{st - M(x, t)\} \quad \text{for all } s \geq 0 \text{ and all } x \in \Omega.$$

M^* is also a Φ -function and is called the complementary function to M in the sense of Young. Moreover, we have the following Young inequality

$$uv \leq M(x, u) + M^*(x, v), \quad \forall u, v \geq 0, \forall x \in \Omega,$$

from which we easily get the Hölder inequality

$$\int_{\Omega} |uv| dx \leq 2\|u\|_{M,\Omega} \|v\|_{M^*,\Omega}$$

for all $u \in L_M(\Omega)$ and $v \in L_{M^*}(\Omega)$. We say that $\{u_k\}_k$ converges to u in norm in $L_M(\Omega)$, if $\|u_k - u\|_{M,\Omega} \rightarrow 0$ as $k \rightarrow \infty$. The notion of the modular convergence is given in the following definition.

Definition 2.1. (Modular convergence) A sequence $\{u_k\}_k$ is said to converge modularly to u in $L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\rho_M((u_k - u)/\lambda) := \int_{\Omega} M(x, |u_k - u|/\lambda) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

For a positive integer m , we define the Musielak–Orlicz–Sobolev spaces $W^m L_M(\Omega)$ and $W^m E_M(\Omega)$ as follows

$$\begin{aligned} W^m L_M(\Omega) &= \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), |\alpha| \leq m\}, \\ W^m E_M(\Omega) &= \{u \in E_M(\Omega) : D^\alpha u \in E_M(\Omega), |\alpha| \leq m\}, \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, $|\alpha| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_N|$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$ stands for the distributional derivatives. Observe that by (1.8) the function $x \rightarrow M^*(x, c)$

always belongs to $L^1_{\text{loc}}(\Omega)$ for every constant number $c \geq 0$. Hence, for an arbitrary compact subset K of Ω we can write by Hölder’s inequality in Musielak spaces

$$\int_K |u(x)| \, dx \leq 2\|u\|_{0,M,\Omega} \|\chi_K\|_{0,M^*,\Omega} \leq 2\left(\int_K M^*(x, 1) \, dx + 1\right) \|u\|_{0,M,\Omega}$$

which yields $L_M(\Omega) \subset L^1_{\text{loc}}(\Omega)$. Therefore, endowed with the Luxemburg norm

$$\|u\|_{m,M,\Omega} = \inf \left\{ \lambda > 0: \sum_{|\alpha| \leq m} \rho_M(D^\alpha u/\lambda) \leq 1 \right\}.$$

$(W^m L_M(\Omega), \|u\|_{m,M,\Omega})$ is a Banach space. We will always identify the space $W^m L_M(\Omega)$ to a subspace of the product $\Pi_{|\alpha| \leq m} L_M = \Pi L_M$.

Definition 2.2. (Segment property) A domain Ω is said to satisfy the segment property, if there exist a finite open covering $\{\theta\}_{i=1}^k$ of $\overline{\Omega}$ and a corresponding non-zero vectors $y_i \in \mathbf{R}^N$ such that $(\overline{\Omega} \cap \theta_i) + ty_i \subset \Omega$ for all $t \in (0, 1)$ and $i = 1, \dots, k$.

This condition holds, for example, if Ω is a bounded Lipschitz domain (cf. [1]). By convention, the empty set satisfies the segment property.

3. Proof of main results

Proof of Theorem 1.1. Let d be the diameter of Ω . As (M2) is concerned and without loss of generality, we can assume that $i = 1$. Being Ω bounded, using a translation if necessary, we may assume that it is contained in the strip $\Omega \subset \{(x_1, x') \in [0, d] \times \mathbf{R}^{N-1}\}$. Let $\partial_1 := \frac{\partial}{\partial x_1}$ stands for the partial derivative operator with respect to x_1 and let us first assume that $u \in C^\infty_0(\Omega)$.

Part 1: We assume that there exists $t_0 \in \mathbf{R}^+$ such that the function $x_1 \in [0, d] \mapsto M((x_1, x'), t)$ changes the variation on both sides of t_0 .

Case 1. Assume that $x_1 \in [0, d] \mapsto M((x_1, x'), t)$ is non-decreasing for $t \leq t_0$ and non-increasing for $t_0 < t$. Defining the two sets

$$E_1 = \left\{ \xi \in [0, d]: |\partial_1 u(\xi, x')| \leq \frac{1}{d} t_0 \right\} \quad \text{and} \quad E_2 = \left\{ \xi \in [0, d]: |\partial_1 u(\xi, x')| > \frac{1}{d} t_0 \right\},$$

we can write

$$\begin{aligned} u(x_1, x') &= u(x_1, x')\chi_{E_1}(x_1) + u(x_1, x')\chi_{E_2}(x_1) \\ &= - \int_{x_1}^d \partial_1(u(\xi, x')\chi_{E_1}(\xi)) \, d\xi + \int_0^{x_1} \partial_1(u(\xi, x')\chi_{E_2}(\xi)) \, d\xi \\ &= - \int_{[x_1,d] \cap E_1} \partial_1 u(\xi, x') \, d\xi + \int_{[0,x_1] \cap E_2} \partial_1 u(\xi, x') \, d\xi. \end{aligned}$$

Thus,

$$|u(x_1, x')| \leq \int_{[x_1,d] \cap E_1} |\partial_1 u(\xi, x')| \, d\xi + \int_{[0,x_1] \cap E_2} |\partial_1 u(\xi, x')| \, d\xi.$$

Then, the convexity of the Φ -function M and Jensen’s inequality enable us to write

$$\begin{aligned} M(x, |u(x_1, x')|) &\leq \frac{1}{2d} \int_0^d M((x_1, x'), 2d |\partial_1 u(\xi, x')| \chi_{[x_1,d] \cap E_1}(\xi)) \, d\xi \\ &\quad + \frac{1}{2d} \int_0^d M((x_1, x'), 2d |\partial_1 u(\xi, x')| \chi_{[0,x_1] \cap E_2}(\xi)) \, d\xi \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2d} \int_{[x_1, d] \cap E_1} M((\xi, x'), 2d |\partial_1 u(\xi, x')|) d\xi \\ &\quad + \frac{1}{2d} \int_{[0, x_1] \cap E_2} M((\xi, x'), 2d |\partial_1 u(\xi, x')|) d\xi \\ &\leq \frac{1}{2d} \int_0^d M((\xi, x'), 2d |\partial_1 u(\xi, x')|) d\xi. \end{aligned}$$

Integrating successively with respect to x' and x_1 , we obtain

$$(3.1) \quad \int_{\Omega} M(x, |u(x)|) dx \leq \frac{1}{2} \int_{\Omega} M(x, 2d |\partial_1 u(x)|) dx.$$

Case 2. Assume that $x_1 \in [0, d] \mapsto M((x_1, x'), t)$ is non-increasing on $t \leq t_0$ and non-decreasing on $t_0 < t$. We can write

$$\begin{aligned} u(x_1, x') &= u(x_1, x') \chi_{E_1}(x_1) + u(x_1, x') \chi_{E_2}(x_1) \\ &= \int_0^{x_1} \partial_1(u(\xi, x') \chi_{E_1}(\xi)) d\xi - \int_{x_1}^d \partial_1(u(\xi, x') \chi_{E_2}(\xi)) d\xi \\ &= \int_{[0, x_1] \cap E_1} \partial_1 u(\xi, x') d\xi - \int_{[x_1, d] \cap E_2} \partial_1 u(\xi, x') d\xi, \end{aligned}$$

which implies

$$|u(x_1, x')| \leq \int_{[0, x_1] \cap E_1} |\partial_1 u(\xi, x')| d\xi + \int_{[x_1, d] \cap E_2} |\partial_1 u(\xi, x')| d\xi.$$

Once again the convexity of the Φ -function M and Jensen's inequality enable us to write

$$\begin{aligned} M(x, |u(x_1, x')|) &\leq \frac{1}{2d} \int_0^d M((x_1, x'), 2d |\partial_1 u(\xi, x')| \chi_{[0, x_1] \cap E_1}(\xi)) d\xi \\ &\quad + \frac{1}{2d} \int_0^d M((x_1, x'), 2d |\partial_1 u(\xi, x')| \chi_{[x_1, d] \cap E_2}(\xi)) d\xi \\ &\leq \frac{1}{2d} \int_{[0, x_1] \cap E_1} M((\xi, x'), 2d |\partial_1 u(\xi, x')|) d\xi \\ &\quad + \frac{1}{2d} \int_{[x_1, d] \cap E_2} M((\xi, x'), 2d |\partial_1 u(\xi, x')|) d\xi \\ &\leq \frac{1}{2d} \int_0^d M((\xi, x'), 2d |\partial_1 u(\xi, x')|) d\xi. \end{aligned}$$

Integrating successively with respect to x' and x_1 , we obtain

$$\int_{\Omega} M(x, |u(x)|) dx \leq \frac{1}{2} \int_{\Omega} M(x, 2d |\partial_1 u(x)|) dx.$$

Part 2: Assume now that for all $t \geq 0$, the function $x_1 \in [0, d] \mapsto M((x_1, x'), t)$ is monotone.

Case 1. Assume first that $x_1 \in [0, d] \mapsto M((x_1, x'), t)$ is non-increasing. By using Jensen's inequality we get

$$\begin{aligned} M(x, |u(x_1, x')|) &\leq M\left((x_1, x'), \int_0^{x_1} |\partial_1 u(\xi, x')| d\xi\right) \\ &\leq M\left((x_1, x'), \int_0^d |\partial_1 u(\xi, x')| \chi_{[0, x_1]}(\xi) d\xi\right) \\ &\leq \frac{1}{d} \int_0^d M((x_1, x'), d|\partial_1 u(\xi, x')| \chi_{[0, x_1]}(\xi)) d\xi \\ &\leq \frac{1}{d} \int_0^{x_1} M((\xi, x'), d|\partial_1 u(\xi, x')|) d\xi \\ &\leq \frac{1}{d} \int_0^d M((\xi, x'), d|\partial_1 u(\xi, x')|) d\xi. \end{aligned}$$

Integrating successively with respect to x' and x_1 , we obtain

$$\int_{\Omega} M(x, |u(x)|) dx \leq \int_{\Omega} M(x, d|\partial_1 u(x)|) dx.$$

Case 2. Assume that $x_1 \in [0, d] \mapsto M((x_1, x'), t)$ is non-decreasing. By virtue of Jensen's inequality we can write

$$\begin{aligned} M(x, |u(x_1, x')|) &\leq M\left((x_1, x'), \int_{x_1}^d |\partial_1 u(\xi, x')| d\xi\right) \\ &\leq M\left((x_1, x'), \int_0^d |\partial_1 u(\xi, x')| \chi_{[x_1, d]}(\xi) d\xi\right) \\ &\leq \frac{1}{d} \int_0^d M((x_1, x'), d|\partial_1 u(\xi, x')| \chi_{[x_1, d]}(\xi)) d\xi \\ &\leq \frac{1}{d} \int_{x_1}^d M((\xi, x'), d|\partial_1 u(\xi, x')|) d\xi \\ &\leq \frac{1}{d} \int_0^d M((\xi, x'), d|\partial_1 u(\xi, x')|) d\xi. \end{aligned}$$

Integrating successively with respect to x' and x_1 , we obtain

$$(3.2) \quad \int_{\Omega} M(x, |u(x)|) dx \leq \int_{\Omega} M(x, d|\partial_1 u(x)|) dx.$$

To sum up, from (3.1)–(3.2), we obtain

$$(3.3) \quad \int_{\Omega} M(x, |u(x)|) dx \leq \int_{\Omega} M(x, 2d|\partial_1 u(x)|) dx,$$

for all $u \in C_0^\infty(\Omega)$.

Let now $u \in W_0^1 L_M(\Omega)$ be arbitrary. By [2, Theorem 3] there exist $\lambda > 0$ and a sequence of functions $u_k \in C_0^\infty(\Omega)$ such that

$$\int_{\Omega} M\left(x, \frac{|u_k(x) - u(x)|}{\lambda}\right) dx + \int_{\Omega} M\left(x, \frac{|\nabla u_k(x) - \nabla u(x)|}{\lambda}\right) dx \rightarrow 0$$

as $k \rightarrow +\infty$. Hence, up to a subsequence still again indexed by k , we can assume that $u_k \rightarrow u$ a.e. in Ω . Then, using (3.3) we can write

$$\begin{aligned} \int_{\Omega} M\left(x, \frac{|u(x)|}{4\lambda d}\right) dx &\leq \liminf_{k \rightarrow +\infty} \int_{\Omega} M\left(x, \frac{|u_k(x)|}{4\lambda d}\right) dx \\ &\leq \liminf_{k \rightarrow +\infty} \int_{\Omega} M\left(x, \frac{1}{2\lambda} |\partial_1 u_k(x)|\right) dx \\ &\leq \frac{1}{2} \liminf_{k \rightarrow +\infty} \int_{\Omega} M\left(x, \frac{1}{\lambda} |\partial_1 u_k(x) - \partial_1 u(x)|\right) dx \\ &\quad + \frac{1}{2} \int_{\Omega} M\left(x, \frac{1}{\lambda} |\partial_1 u(x)|\right) dx \\ &\leq \frac{1}{2} \int_{\Omega} M\left(x, \frac{1}{\lambda} |\partial_1 u(x)|\right) dx. \end{aligned}$$

Thus, (1.9) is proved. Let us now prove the inequality (1.10). For $u \in W_0^m L_M(\Omega)$, it can be checked easily from (1.9) that

$$\sum_{|\alpha| < m} \int_{\Omega} M\left(x, \frac{|D^\alpha u(x)|}{C(m, \Omega) \sum_{|\beta|=m} \|D^\beta u\|_{M, \Omega}}\right) dx \leq 1,$$

where $C(m, \Omega) = c_{m, \Omega} \left(1 + \sum_{|\beta|=m} 1\right)$ depending only on m and Ω . The proof of Theorem 1.1 is then achieved. \square

Proof of Theorem 1.2. Let $u \in W_0^m E_M(\Omega)$. By the definition of $W_0^m E_M(\Omega)$, there exists a sequence $\{u_k\}_k$ of $\mathcal{C}_0^\infty(\Omega)$ functions such that $D^\alpha u_k \rightarrow D^\alpha u$ for all $|\alpha| \leq m$ with respect to the norm topology in $L_M(\Omega)$. As the norm convergence implies the modular one, one has $D^\alpha u_k \rightarrow D^\alpha u$ for all $|\alpha| \leq m$ with respect to the modular topology. Therefore, we get the result by following exactly the same lines of the proof of Theorem 1.1. \square

Proof of Theorem 1.3. The embedding $W_0^m E_M(\Omega) \subset K_0^m L_M(\Omega)$ is obviously satisfied. It only remains to show that $K_0^m L_M(\Omega) \subset W_0^m E_M(\Omega)$ holds true. Let $u \in K_0^m L_M(\Omega)$ and let $\eta > 0$ be arbitrary. We will show that there is a sequence $v \in \mathcal{C}_0^\infty(\Omega)$ such

$$(3.4) \quad \sum_{|\alpha| \leq m} \|D^\alpha u - D^\alpha v\|_{M, \Omega} \leq \eta.$$

By the definition of $K_0^m L_M(\Omega)$ there exist a sequence $\{u_k\}$ in $W^m L_M(\Omega)$ of compactly supported functions in Ω and $k_\alpha > 0$ such that for all $k > k_\alpha$ and $|\alpha| \leq m$ we have

$$\|D^\alpha u - D^\alpha u_k\|_{M, \Omega} \leq \frac{\eta}{2K}.$$

where K is the total number of multi-indices with $|\alpha| \leq m$. Now by using [2, Lemma 12] there exist a sequence $\{u_k^n\}$ in $\mathcal{C}_0^\infty(\Omega)$ and $n_\eta^{k, \alpha} > 0$ such that for all $n > n_\eta^{k, \alpha}$ and $|\alpha| \leq m$ we have

$$\|D^\alpha u_k - D^\alpha u_k^n\|_{M, \Omega} \leq \frac{\eta}{2K}.$$

By the triangle inequality we get for all $k > k_0^\alpha$ and $n > n_\eta^{k, \alpha}$

$$\|D^\alpha u - D^\alpha u_k^n\|_{M, \Omega} \leq \|D^\alpha u - D^\alpha u_k\|_{M, \Omega} + \|D^\alpha u_k - D^\alpha u_k^n\|_{M, \Omega} \leq \frac{\eta}{K},$$

Hence follows (3.4) and then the following identification

$$K_0^m L_M(\Omega) = W_0^m E_M(\Omega).$$

Therefore, Theorem 1.3 follows immediately from Theorem 1.2. □

Proof of Theorem 1.4. We begin first by showing that $W_0^m L_M(\Omega) \subset W_0^{m,1}(\Omega) \cap W^m L_M(\Omega)$. Since $W_0^m L_M(\Omega)$ is a subset of $W^m L_M(\Omega)$, it's sufficient to check that for all function u belonging to $W_0^m L_M(\Omega)$ we have $u \in W_0^{m,1}(\Omega)$. Let $u \in W_0^m L_M(\Omega)$. By [2, Theorem 3] there exist $\lambda > 0$ and a sequence $u_k \in C_0^\infty(\Omega)$ such that

$$\sum_{|\alpha| \leq m} \int_{\Omega} M(x, |D^\alpha u_k(x) - D^\alpha u(x)|/\lambda) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore, for all $|\alpha| \leq m$

$$\int_{\Omega} M(x, |D^\alpha u_k(x) - D^\alpha u(x)|/\lambda) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Thus, for a subsequence still denoted by u_k , we can assume

$$D^\alpha u_k \rightarrow D^\alpha u \quad \text{a.e. in } \Omega.$$

Applying Vitali's theorem we obtain

$$\int_{\Omega} |D^\alpha u_k(x) - D^\alpha u(x)| dx \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which implies that $u \in W_0^{m,1}(\Omega)$.

Conversely, we should prove that $W_0^{m,1}(\Omega) \cap W^m L_M(\Omega) \subset W_0^m L_M(\Omega)$. We will show that for $u \in W_0^{m,1}(\Omega) \cap W^m L_M(\Omega)$ there exist a sequence $v \in C_0^\infty(\Omega)$ such that v converges in the modular sense to u in $W^m L_M(\Omega)$ and then we conclude by using [2, Theorem 3]. Let us denote by \tilde{u} the extension of u by zero outside Ω . Since u belongs to $W_0^{m,1}(\Omega)$ it yields, $\tilde{u} \in W^{m,1}(\mathbf{R}^N)$ and $\widetilde{D^\alpha u} = D^\alpha \tilde{u}$ in the distributional sense and a.e. in \mathbf{R}^N (see [1, Lemma 3.27]) and so $\tilde{u} \in W^m L_M(\mathbf{R}^N)$. Then by [2, Lemma 3] we can assume that u has compact support $K \subset \bar{\Omega}$. We will distinguish the two cases: either $K \subset \Omega$ or $K \cap \partial\Omega \neq \emptyset$. If $K \subset \Omega$ then we get the desired inclusion by [2, Lemma 12]. If $K \cap \partial\Omega \neq \emptyset$, then, as in the proof of [2, Theorem 2], there exist a finite collection $\{\hat{\theta}_i\}_{i=1}^k$ covering the compact set $K \cap \partial\Omega$ and an open covering $\{\theta'_i\}_{i=0}^k$ of K with θ'_i has a compact closure in $\hat{\theta}_i$ for $i = 0, 1, \dots, k$. Then u can be splitted into finitely-many pieces u_i , such that $u = \sum_{i=1}^k u_i$ with $\text{supp } u_i \subset \theta'_i$, $i = 0, 1, \dots, k$.

For $i = 0$, we consider $\text{supp } u_0 \subset \theta'_0 \subset \Omega$, then as for the first case by [2, Lemma 12] there exist $\varepsilon_0 > 0$ small enough ($\varepsilon_0 < \text{dist}(\theta'_0, \partial\Omega)$), such the regularized function $v_0 = J_{\varepsilon_0} * u_0$ belongs to $C_0^\infty(\Omega)$ and converge in modular since to u in $W^m L_M(\Omega)$. For $1 \leq i \leq k$ fixed. Let z_i be a non-zero vector associated to $\hat{\theta}_i$ by the segment property and let $r_i \in (0, 1)$ be such that

$$0 < r_i < \min\{1/(|z_i| + 1), \text{dist}(\theta'_i, \partial\hat{\theta}_i)|z_i|^{-1}\}.$$

Define

$$(u_i)_{-r_i}(x) = u_i(x - r_i z_i).$$

and choose

$$\varepsilon_i < \text{dist}((\theta'_i \cap \bar{\Omega}) + r_i z_i, \mathbf{R}^N \setminus \Omega).$$

We define then the sequences

$$v_i^{\varepsilon_i, r_i}(x) = J_{\varepsilon_i} * (u_i)_{-r_i} = \int_{B(0,1)} J(y) u_i(x - r_i z_i - \varepsilon_i y) dy$$

and

$$v(x) = \sum_{i=1}^k v_i^{\varepsilon_i, r_i}(x) + J_{\varepsilon_0} * u_0(x),$$

therefore, $v \in C_0^\infty(\Omega)$. Then, arguing similarly as in the proof of [2, Theorem 2], we prove that v converges to u in $W^m L_M(\Omega)$ with respect to the modular convergence. This implies that u belongs to $W_0^m L_M(\Omega)$.

To check (1.11) observe that $\{u \in W^1 L_M(\Omega) : \text{tr}(u) = 0 \text{ on } \partial\Omega\} \subset \{u \in W^{1,1}(\Omega) : \text{tr}(u) = 0 \text{ on } \partial\Omega\} = W_0^{1,1}(\Omega)$. So that for any $v \in \{u \in W^1 L_M(\Omega) : \text{tr}(u) = 0 \text{ on } \partial\Omega\}$ one has $v \in W_0^{1,1}(\Omega) \cap W^1 L_M(\Omega) = W_0^1 L_M(\Omega)$. \square

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