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Area operators on large Bergman spaces

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Abstract. We completely characterize those positive Borel measures μ on the open unit disk $\mathbb{D}$ for which the area operator $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded. Here, the indices $0 < p, q < \infty$ are arbitrary and φ belongs to a certain class $\mathcal{W}_0$ of exponentially decreasing weights. Accordingly, the proofs require techniques adapted to such weights, like tent spaces, Carleson measures for A^p_φ-spaces, Kahane–Khintchine inequalities, and decompositions of the unit disc by (ρ, r)-lattices, which differ from the conventional decompositions into subsets with essentially constant hyperbolic radii.

Pinta-alaoperaattorit suurissa Bergman-avaruuksissa

Tiivistelmä. Karakterisoimme ne avoimen yksikkökiekon $\mathbb{D}$ positiiviset Borel-mitat μ, joille pinta-alaoperaattori $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ on rajoitettu. Indeksit $0 < p, q < \infty$ voivat tässä olla mielivaltaiset, ja funktio φ kuuluu tiettyyn eksponentiaalisesti laskevien painojen luokkaan $\mathcal{W}_0$. Todistuksissa tarvitaan tällaisiin painoihin soveltuvia tekniikoita, kuten teltta-avaruuksia, A^p_φ-avaruuksien Carleson-mittoja, Kahane–Khintchine-epäyhtälöitä sekä yksikkökiekon hajotelmia (ρ, r)-hiloihin, jotka poikkeavat tavanomaisista, hyperboliselta halkaisijaltaan vakiomittaisista joukoista koostuvista hajotelmista.

1. Introduction

Given a positive Borel measure μ on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, an *area operator* A_μ is the sublinear operator defined by

$$A_\mu(f)(\zeta) = \int_{\Gamma(\zeta)} |f(z)| \frac{d\mu(z)}{1-|z|}, \tag{1.1}$$

where f is a holomorphic function on $\mathbb{D}$, $\zeta \in \mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$ and $\Gamma(\zeta)$ is the Stolz angle (see below for definition). The importance of area operators stems from their apparent connections to nontangential maximal functions, Littlewood–Paley operators, multipliers, Poisson integrals, tent spaces and so on. The purpose of this paper is to characterize those measures μ for which A_μ is a well-defined and bounded operator $A^p_\varphi \to L^q(\mathbb{T})$, where $0 < p, q < \infty$ and A^p_φ is a weighted Bergman space on $\mathbb{D}$ with φ belonging to a weight class $\mathcal{W}_0$.

We recall that in the setting of Hardy spaces, the boundedness of A_μ from H^p to $L^q(\mathbb{T})$ was studied by Cohn [6] in the case $0 < p = q < \infty$ and by Gong, Lou and Wu [9] in the cases $0 < p \le q < \infty$ and $1 \le q < p < \infty$. As for Bergman spaces, Wu [17] discussed the boundedness of A_μ from A^p_α to $L^q(\mathbb{T})$ for $0 < p \le q < \infty$

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and $1 \le q < p < \infty$. Here, A^p_α, $\alpha > -1$, denotes the weighted Bergman space with standard radial power weight. The first named author [1] extended this topic to the Bergman spaces with exponential weights ω introduced by Borichev, Dhuez and Kellay in [5]. He provided a sufficient condition for the boundedness of $A_\mu\colon A^p_\omega \to L^q(\mathbb{T})$, where $0 < p \le q < \infty$ or $1 \le q < p < \infty$, and left the necessity open. By for example [10], the class of the exponential weights in [5] is a subset of he class $\mathcal{W}_0$, which, as mentioned, is the weight class to be treated in this paper. Finally, it is worth mentioning that the second named author together with Pau and Wang [15] characterized the bounded area operators on the Bergman spaces in the unit ball of $\mathbb{C}^N$, $N > 1$.

Let us next introduce the necessary notation and definitions. First, we write $H(\mathbb{D})$ for the space of all analytic functions in $\mathbb{D}$. Given p with $0 < p < \infty$ and a positive Borel measure μ on $\mathbb{D}$, we denote by $L^p_\mu = L^p(\mathbb{D}, d\mu)$ the space of p-integrable functions with respect to the measure μ on $\mathbb{D}$. In the case μ equals the normalized Lebesgue area measure on $\mathbb{D}$, i.e., $d\mu = \pi^{-1}\,dx\,dy = dA$, the spaces are denoted simply by L^p. Also, L^∞ stands for the Banach space of bounded, measurable functions on $\mathbb{D}$, endowed with the standard unweighted sup-norm. Recall that for $0 < p < 1$, L^p_μ is only a quasi-Banach space, i.e., a vector space with a quasi-norm defining a complete metrizable topology. (A quasi-norm $\|\cdot\|$ in a vector space X satisfies the norm axioms except that instead of the triangle inequality there only holds

$$\|x + y\| \le C(\|x\| + \|y\|) \quad \text{for a constant } C \ge 1, \text{ for all } x, y \in X.)$$

Given $0 < p < \infty$, the Banach or quasi-Banach space of p-summable sequences of complex numbers is denoted in a standard way by ℓ^p, and ℓ^∞ denotes the Banach space of bounded sequences. Also, H^p with $0 < p < \infty$ stands for the standard Hardy spaces on $\mathbb{D}$ or $\mathbb{T}$. Given any (quasi-)Banach space X, its (quasi-)norm is denoted by $\|\cdot\|_X$. Given a quasi-Banach space X, a mapping $F\colon X \to X$ is a sublinear operator, if $\|F(x+y)\|_X \le \|F(x)\|_X + \|F(y)\|_X$ and $\|F(\lambda x)\|_X = |\lambda|\|F(x)\|_X$ for all $x, y \in X$ and all scalars λ. Such an operator is bounded, if there exists a constant $C > 0$ such that $\|F(x)\|_X \le C\|x\|_X$ for all $x \in X$.

Let $\mathcal{C}_0$ be the space of all continuous real valued functions on $\mathbb{D}$ that vanish at the boundary of $\mathbb{D}$. We denote

$$\mathcal{L} = \left\{ \rho \in \mathcal{C}_0 \colon \|\rho\|_{\mathcal{L}} = \sup_{z,w\in\mathbb{D}, z\ne w} \frac{|\rho(z) - \rho(w)|}{|z - w|} < \infty \right\},$$

and the class $\mathcal{L}_0$ is defined to consist of those $\rho \in \mathcal{L}$ with the property that for each $\varepsilon > 0$, there exists a compact subset $E \subset \mathbb{D}$ such that $|\rho(z) - \rho(w)| \le \varepsilon|z - w|$, whenever $z, w \in \mathbb{D}\backslash E$. We also write

$$\mathcal{W}_0 = \left\{ \varphi \in C^2 \colon \Delta\varphi > 0, \text{ and } \exists \rho \in \mathcal{L}_0 \text{ such that } \frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \right\},$$

where Δ denotes the standard Laplace operator. Here and later, given some functions F and G with positive values, the notation $F \asymp G$ indicates that there exists some positive constant C, in particular independent of the variables of F and G, such that $C^{-1}F \le G \le CF$. Similarly, we write $F \lesssim G$ if there exists a constant $C > 0$ such that $F \le CG$.

Given p with $0 < p < \infty$ and a subharmonic function φ on $\mathbb{D}$, the exponential type weighted Bergman space A^p_φ consists of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\varphi} = \left(\int_{\mathbb{D}} \left|f(z)e^{-\varphi(z)}\right|^p dA(z)\right)^{\frac{1}{p}} < \infty.$$

We will focus on Bergman spaces A^p_φ induced by the weights $w = e^{-p\varphi}$ with $\varphi \in \mathcal{W}_0$. These spaces were introduced by Hu, the second named author and Schuster in [10]. It follows from Lemma 3.3 in [10] that there exists a reproducing kernel $K_z(\cdot) = K(\cdot, z)$ in A^2_φ having the property

$$f(z) = \int_{\mathbb{D}} f(w)K(z,w)e^{-2\varphi(w)}\, dA(w)$$

for all $f \in A^2_\varphi$ and $z \in \mathbb{D}$. We denote by $\kappa_{p,z} = K_z/\, \|K_z\|_{A^p_\varphi}$ the normalized Bergman kernel of A^p_φ.

Let $D(z,r) \subset \mathbb{C}$ be the Euclidean disc with center at z and radius $r > 0$. For simplicity, we write $D^r(z)$ for the disc $D(z, r\rho(z))$. Given a positive Borel measure μ on $\mathbb{D}$ and $t, r > 0$, the *general Berezin transform* $\widetilde{\mu}_t$ of μ is defined by

$$\widetilde{\mu}_t(z) = \int_{\mathbb{D}} |\kappa_{t,z}(w)|^t\, e^{-t\varphi(w)}\, d\mu(w), \quad z \in \mathbb{D},$$

and the *general averaging function* $\widehat{\mu}_{r,p}$ by

$$\widehat{\mu}_{r,p}(z) = \frac{\int_{D^r(z)} e^{p\varphi(w)} d\mu(w)}{\rho(z)^2}, \quad z \in \mathbb{D}.$$

In particular, the classical averaging function is $\widehat{\mu}_r(z) = \mu(D^r(z))/\rho(z)^2, z \in \mathbb{D}$.

If $\zeta \in \mathbb{T}$ and $\gamma > 2$ are given, the *Stolz angle* $\Gamma(\zeta)$ with aperture $\gamma/2$ is defined by

$$\Gamma_\gamma(\zeta) = \left\{ z \in \mathbb{D}\colon |\zeta - z| < \frac{\gamma}{2}(1 - |z|) \right\}.$$

In this paper we denote $\Gamma(\zeta) := \Gamma_4(\zeta)$, but Stolz angles with other apertures will also be used. Now, recall that if μ is a positive Borel measure on $\mathbb{D}$, the area operator A_μ acting on $H(\mathbb{D})$ is the sublinear operator defined by the formula (1.1).

For every $z \in \mathbb{D}$, let us denote

$$I(z) = \{\zeta \in \partial\mathbb{D}\colon z \in \Gamma(\zeta)\}.$$

It is clear that $I(z)$ is an open arc on $\partial\mathbb{D}$ with center $z/|z|$ whenever $z \neq 0$. Moreover, $|I(z)| \asymp 1 - |z|$. It is also easy to see that for any open arc $I \subset \partial\mathbb{D}$, there exists a $z \in \mathbb{D}$ such that $I(z) = I$. Also, for every open arc $I \subset \partial\mathbb{D}$ the set

$$S(I) = \left\{ z \in \mathbb{D}\colon \frac{z}{|z|} \in I,\ 1 - |I| \leqslant |z| \right\}$$

is called the *Carleson square* based on I. Finally, for a positive function g defined on the unit circle $\mathbb{T}$, we write

$$Tg(z) = \frac{1}{1-|z|} \int_{I(z)} g(\lambda)|d\lambda|, \quad z \in \mathbb{D}.$$

By Theorem 2.4 in [17], there holds $Tg(z) \leq CPg(z)$ for all $z \in \mathbb{D}$, where Pg is the Poisson integral of g.

With these preparations we are ready to formulate our main result, which contains characterizations of the boundedness of the area operators acting in Bergman spaces with exponential weights. The proof will be given in Section 3 after some preliminary considerations in Section 2.

Main Theorem. *Let the functions φ and ρ on $\mathbb{D}$ be such that $\varphi \in \mathcal{W}_0$ and $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$ and let μ be a finite positive Borel measure on $\mathbb{D}$.*

(i) *If $1 < p \le q < \infty$, then $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if $\widehat{\mu}_{\delta,1}(z)^{p'}\, dA(z)$ is a p'/q'-Carleson measure for some (or any) small enough $\delta \in (0,\alpha]$.*

(ii) *If $p \le \min\{1,q\}$, then the area operator $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if*

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|)^{\frac{1-q}{q}}}{\rho(w)^{(2-2p)/p}} \widehat{\mu}_{r,1}(w) < \infty$$

for every sufficiently small $r > 0$.

(iii) *If $1 \le q < p < \infty$, then $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if*

$$G(\zeta) := \int_{\Gamma(\zeta)} \frac{\widehat{\mu}_{\delta,1}(z)^{p'}}{1-|z|}\, dA(z) \in L^{\frac{q(p-1)}{p-q}}(\mathbb{T})$$

for some (or any) sufficiently small $\delta \in (0,\alpha]$.

(iv) *If $0 < q < p \le 1$, then $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if for every small enough $r > 0$ there exists a Stolz angle $\widetilde{\Gamma}(\zeta)$ with vertex at ζ and aperture bigger than the aperture of $\Gamma(\zeta)$ such that the function*

$$G^\mu_{p,q}(\zeta) := \sup_{w\in\widetilde{\Gamma}(\zeta)} \frac{(1-|w|)^{\frac{1-p}{p}}}{\rho(w)^{(2-2p)/p}} \widehat{\mu}_{r,1}(w)$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{T})$.

(v) *If $0 < q < 1 \le p$, then $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if*

$$M^\mu_{p,q}(w) := \int_{\widetilde{\Gamma}(\zeta)} \frac{(1-|w|)^{\frac{1-p}{p}}}{\rho(w)^{2/p}} \widehat{\mu}_{r,1}(w)\, dA(w)$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{T})$ for every sufficiently small $r > 0$.

2. Preliminaries

In this section, we present a number of preliminary results which will be used in the arguments in Section 3. The proofs of these statements can mostly be found in the existing literature.

2.1. Kahane–Khinchine inequalities. Let us start by the classical Khinchine's inequality; see Appendix A in [8] for more details. For all $k \in \mathbb{N} = \{1,2,\ldots\}$ we denote by $r_k\colon [0,1] \to \{0,\pm 1\}$, $r_k(t) = \operatorname{sign}\sin(2^k\pi t)$, the kth Rademacher function.

Khinchine's inequality: Let $0 < p < \infty$. Then,

$$\left(\sum_k |c_k|^2\right)^{p/2} \asymp \int_0^1 \left|\sum_k c_k r_k(t)\right|^p dt, \tag{2.1}$$

where $\{c_k\}_{k=1}^\infty$ is an arbitrary sequence of complex scalars.

Next, we recall Kahane's inequality. The details can be found in [12] and [13].

Kahane's inequality: Let X be a quasi-Banach space and let $0 < p, q < \infty$. There holds

$$\left(\int_0^1 \left\|\sum_k r_k(t)x_k\right\|_X^q dt\right)^{1/q} \asymp \left(\int_0^1 \left\|\sum_k r_k(t)x_k\right\|_X^p dt\right)^{1/p}, \tag{2.2}$$

where $\{x_k\}_{k=1}^\infty$ is an arbitrary sequence in X. Moreover, the implicit constants in (2.2) depend only on p and q and not on the quasi-Banach space X.

2.2. Separated sequences and (ρ, r)-lattices. We denote by $\beta\colon \mathbb{D}\times\mathbb{D} \to \mathbb{R}_0^+ = [0,\infty)$ the Bergman metric on $\mathbb{D}$; see [18], Section 4.2. for the definition. Let $D(a,r) = \{z \in \mathbb{D}\colon \beta(a,z) < r\}$ be the hyperbolic disk of radius $r > 0$ centered at $a \in \mathbb{D}$. A sequence of points $Z = \{z_k\}_{k=1}^\infty \subset \mathbb{D}$ is said to be *separated* if there exists $\delta > 0$ such that $\beta(z_k, z_j) \geq \delta$ for all $k, j \in \mathbb{N}$ with $k \neq j$. This implies that there exists $r > 0$ such that the hyperbolic disks $D(z_k, r)$, $k \in \mathbb{N}$, are mutually disjoint.

Decompositions of the unit disc into subsets with approximately constant hyperbolic radii are standard tools in the Bergman space theory, see for example Section 4.2. of [18]. Such decompositions are however quite useless in the case of Bergman spaces with exponentially decreasing weights, but they can be replaced by decompositions with varying hyperbolic radii. The related results will be needed later, and they are contained in the following Lemmas 2.1-2.5, the proofs of which can be found in [2, 3, 10].

Lemma 2.1. *Let $\rho \in \mathcal{L}$ be a positive function. Then, there exists a constant $\alpha_1 > 0$ such that the following holds.*

(a) *There exist constants $C_1, C_2 > 0$ such that*

$$C_1\rho(w) \leq \rho(z) \leq C_2\rho(w),$$

for all $z \in \mathbb{D}$ and $w \in D^{\alpha_1}(z)$.

(b) *There exists a constant $B > 0$ such that for all $z \in \mathbb{D}$,*

$$D^r(z) \subset D^{Br}(w), \quad D^r(w) \subset D^{Br}(z), \tag{2.3}$$

for all $w \in D^r(z)$ and $0 < r \leq \alpha_1$.

If α_1 and B are as in Lemma 2.1, it follows that there exists an $s > 0$ such that for $0 < r \leq \alpha_1$ there exists a sequence $\{z_k\}_{k=1}^\infty \subset \mathbb{D}$ with the following properties (recall the notation $D^r(z) = D(z, r\rho(z))$):

(a) $\mathbb{D} = \bigcup_{k=1}^\infty D^r(z_k)$,

(b) $D^{sr}(z_k) \cap D^{sr}(z_j) = \emptyset$ for all $k \neq j$,

(c) there exists a positive integer N depending only on B, r such that

$$1 \leq \sum_{k=1}^\infty \chi_{D^{Br}(z_k)}(z) \leq N$$

for all $z \in \mathbb{D}$, where χ_E is the characteristic function of a set E.

A sequence $\{z_k\}_{=1}^\infty$ with properties (a)–(c) is called a (ρ, r)*-lattice*. Obviously, every (ρ, r)-lattice is a separated sequence.

Let us next consider the following subharmonicity property.

Lemma 2.2. *Assume $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$ and $0 < p < \infty$. Then, there exist constants $\alpha_2 > 0$ and $C > 0$ such that*

$$\left|f(z)e^{-\varphi(z)}\right|^p \leq C\frac{1}{\rho(z)^2}\int_{D^r(z)} \left|f(w)e^{-\varphi(w)}\right|^p dA(w),$$

for $r \in (0, \alpha_2]$ and $f \in H(\mathbb{D})$.

We now fix, for the rest of this paper, the constant α to be the smallest of the numbers α_1 and α_2 in the previous lemmas.

The following kernel estimates can be found in [10].

Lemma 2.3. *Assume that $0 < p < \infty$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$. Then, for all $w \in \mathbb{D}$ and $r \in (0, \alpha]$, there holds*

$$|\kappa_w(z)|\, e^{-\varphi(z)} \asymp \frac{1}{\rho(w)}, \quad z \in D^r(w),$$

and, for any fixed positive constant N,

$$|\kappa_w(z)|\, e^{-\varphi(z)} \lesssim \frac{1}{\rho(z)} \left(\frac{\min\{\rho(z), \rho(w)\}}{|z-w|} \right)^N, \quad z \in \mathbb{D},$$

Lemma 2.4. *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$ and $0 < p < \infty$. We have*

$$\|K_z\|_{A^p_\varphi} \asymp e^{\varphi(z)} \rho(z)^{\frac{2}{p}-2}, \quad z \in \mathbb{D}$$

and

$$\|\kappa_z\|_{A^p_\varphi} \asymp \rho(z)^{\frac{2}{p}-1}, \quad z \in \mathbb{D}.$$

Finally, we will need the test functions provided by the next lemma.

Lemma 2.5. *Let $\{z_k\}_{k=1}^\infty$ be a (ρ, r)-lattice, and let $0 < r \le \alpha$ and $0 < p < \infty$. Given a sequence $\lambda = \{\lambda_k\}_{k=1}^\infty \in \ell^p$, set*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k \kappa_{z_k}(z) \rho(z_k)^{1-\frac{2}{p}}, \quad z \in \mathbb{D}.$$

Then $f \in A^p_\varphi$ and $\|f\|_{p,\varphi} \le C \|\lambda\|_{\ell^p}$.

2.3. Tent spaces. Tent spaces were first introduced by Coifman, Meyer and Stein [7] in order to study certain problems in harmonic analysis, and they form a general framework for questions regarding classical spaces of analytic functions, including Hardy, Bergman and BMOA spaces among others.

Let $0 < p, q < \infty$ and let $Z = \{z_k\}_{k=1}^\infty$ be a separated sequence. The tent sequence space $T^p_q(Z)$ consists of complex sequences $\lambda = \{\lambda_k\} = \{\lambda_k\}_{k=1}^\infty$ satisfying

$$\|\lambda\|^p_{T^p_q(Z)} := \int_{\mathbb{T}} \left(\sum_{\{k : z_k \in \Gamma(\zeta)\}} |\lambda_k|^q \right)^{\frac{p}{q}} |d\zeta| < \infty.$$

Also, $\lambda = \{\lambda_k\} \in T^p_\infty(Z)$, if

$$\|\lambda\|^p_{T^p_\infty(Z)} := \int_{\mathbb{T}} \Big(\sup\{|\lambda_k| \colon z_k \in \Gamma(\zeta)\} \Big)^p |d\zeta| < \infty.$$

Finally, $\lambda = \{\lambda_k\} \in T^\infty_q(Z)$, if

$$\|\lambda\|_{T^\infty_q(Z)} = \sup_{\zeta \in \mathbb{T}} \left(\sup_{w \in \Gamma(\zeta)} \frac{1}{(1-|w|^2)} \sum_{\{k : z_k \in I(w)\}} |\lambda_k|^q \left(1 - |z_k|^2\right) \right)^{1/q} < \infty.$$

It is well known that $\lambda \in T^\infty_q(Z)$ if and only if $\mu_\lambda = \sum_{k=1}^\infty |\lambda_k|^q \left(1-|z_k|^2\right) \delta_{z_k}$ is a Carleson measure, where δ_a denotes the Dirac point mass at the point a.

2.4. Carleson measures. Let μ be a finite positive Borel measure on $\mathbb{D}$ and let $0 < p, q < \infty$. We recall that μ is a q-Carleson measure for A^p_φ if the identity operator $\mathrm{Id}\colon A^p_\varphi \to L^q_\varphi(\mu)$ is bounded, where $L^q_\varphi(\mu)$ consists of all μ-measurable functions f on $\mathbb{D}$ for which

$$\|f\|_{L^q_\varphi(\mu)} = \left(\int_{\mathbb{D}} |f(z)|^q e^{-q\varphi(z)}\, d\mu(z)\right)^{1/q} < \infty.$$

Correspondingly, μ is a vanishing q-Carleson measure for A^p_φ if the identity operator $\mathrm{Id}\colon A^p_\varphi \to L^q_\varphi(\mu)$ is compact, i.e.

$$\lim_{j\to\infty} \int_{\mathbb{D}} |f_j(z)|^q\, e^{-q\varphi(z)}\, d\mu(z) = 0$$

whenever $\{f_j\}_{j=1}^\infty$ is a bounded sequence in A^p_φ that converges to 0 uniformly on any compact subset of $\mathbb{D}$ as $j \to \infty$.

The next Lemmas 2.6 and 2.7 can be found in [1].

Lemma 2.6. *Let $0 < p \le q < \infty$. Then the embedding $\mathrm{Id}\colon H^p \to L^q(\mu)$ is bounded if and only if μ is an q/p-Carleson measure.*

Lemma 2.7. *Let $0 < q < p < \infty$. The following conditions are equivalent.*

(a) $\mathrm{Id}\colon H^p \to L^q(\mu)$ *is bounded.*
(b) *The function $A_\mu 1(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|}$ belongs to $L^{\frac{p}{p-q}}(\mathbb{T})$.*
(c) *The sweep $\breve{\mu}$ of μ, defined by*

$$\breve{\mu}(\zeta) = \frac{1}{2\pi}\int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(z),$$

belongs to $L^{\frac{p}{p-q}}(\mathbb{T})$.

Lemmas 2.8 and 2.9 are given by Theorems 2.6 and 2.8 in [14].

Lemma 2.8. *Let $0 < p \le q < \infty, \varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$ and μ be a finite positive Borel measure on $\mathbb{D}$. Then, the following statements are equivalent:*

(a) *μ is a q-Carleson measure for A^p_φ;*
(b) *$\widetilde{\mu}_t \rho^{2-2q/p} \in L^\infty$;*
(c) *$\widehat{\mu}_\delta \rho^{2-2q/p} \in L^\infty$ for some (or any) small enough $\delta \in (0, \alpha]$;*
(d) *we have*

$$\left\{\widehat{\mu}_r\left(z_k\right) \rho\left(z_k\right)^{2-2q/p}\right\}_{k=1}^\infty \in \ell^\infty$$

for some (or any) (ρ, r)-lattice $\{z_k\}$ with a small enough $r \in (0, \alpha]$. Moreover, there holds

$$\begin{aligned}\|Id\|^q_{A^p_\varphi \to A^q_\varphi} &\asymp \left\|\widetilde{\mu}_t \rho^{2-2q/p}\right\|_{L^\infty} \asymp \left\|\widehat{\mu}_\delta \rho^{2-2q/p}\right\|_{L^\infty} \\ &\asymp \left\|\left\{\widehat{\mu}_r\left(z_k\right) \rho\left(z_k\right)^{2-2q/p}\right\}_{k=1}^\infty\right\|_{\ell^\infty}.\end{aligned}$$

Lemma 2.9. *Let $0 < q < p < \infty$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$, and assume μ is a finite positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(a) *μ is a vanishing q-Carleson measure for A^p_φ;*
(b) *μ is a q-Carleson measure for A^p_φ;*
(c) *$\widetilde{\mu}_t \in L^{\frac{p}{p-q}}$;*
(d) *$\widehat{\mu}_\delta \in L^{\frac{p}{p-q}}$ for some (or any) $\delta \in (0, \alpha]$ small enough;*

(e) *we have*

$$\left\{\widehat{\mu}_r(z_k)\,\rho(z_k)^{2-2q/p}\right\}_{k=1}^{\infty} \in \ell^{\frac{p}{p-q}}$$

for some (or any) (ρ, r)*-lattice* $\{z_k\}_k$ *with* $r \in (0, \alpha]$. *Moreover, there holds*

$$\|Id\|_{A^p_\varphi \to A^q_\varphi} \asymp \|\widetilde{\mu}_t\|_{L^{\frac{p}{p-q}}} \asymp \|\widehat{\mu}_\delta\|_{L^{\frac{p}{p-q}}}$$
$$\asymp \left\| \left\{\widehat{\mu}_r(z_k)\,\rho(z_k)^{2-2q/p}\right\}_{k=1}^{\infty} \right\|_{\ell^{\frac{p}{p-q}}}.$$

2.5. Additional results. We will need the following duality results for tent sequence spaces. See [4], [11] and [13] for the details of the proofs. Given $1 < p < \infty$ we denote the dual index by $p' = p/(p-1)$ in the sequel.

Lemma 2.10. *Let* $1 < p < \infty$ *and* $Z = \{z_k\}$ *be a separated sequence. If* $1 < q < \infty$, *then the dual of* $T^p_q(Z)$ *is isomorphic to* $T^{p'}_{q'}(Z)$ *under the pairing*

$$\langle \lambda, \mu \rangle_{T^2_2(Z)} = \sum_k \lambda_k \overline{\mu_k}\left(1 - |z_k|^2\right), \quad \text{where } \lambda \in T^p_q(Z), \quad \mu \in T^{p'}_{q'}(Z).$$

If $0 < q \le 1$, *then the dual of* $T^p_q(Z)$ *is isomorphic to* $T^{p'}_\infty(Z)$ *under the same pairing.*

The following factorization of tent sequence spaces was proved by Miihkinen, Pau, Perälä and Wang in [16].

Lemma 2.11. *Assume that* $0 < p, q < \infty$ *and that* $Z = \{z_k\}$ *is an* r*-lattice. Let* $p < p_1,\ p_2 < \infty$, $q < q_1,\ q_2 < \infty$ *satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then, we have

$$T^p_q(Z) = T^{p_1}_{q_1}(Z) \cdot T^{p_2}_{q_2}(Z);$$

in other words, if $\gamma = \{\gamma_k\} \in T^{p_1}_{q_1}(Z)$ *and* $\beta = \{\beta_k\} \in T^{p_2}_{q_2}(Z)$, *then* $\gamma \cdot \beta = \{\gamma_k \beta_k\}_{k=1}^{\infty} \in T^p_q(Z)$ *with* $\|\gamma \cdot \beta\|_{T^p_q(Z)} \lesssim \|\gamma\|_{T^{p_1}_{q_1}(Z)} \cdot \|\beta\|_{T^{p_2}_{q_2}(Z)}$.

Conversely, if $\lambda \in T^p_q(Z)$, *then there are sequences* $\gamma \in T^{p_1}_{q_1}(Z)$ *and* $\beta \in T^{p_2}_{q_2}(Z)$ *such that* $\lambda = \gamma \cdot \beta$ *and* $\|\gamma\|_{T^{p_1}_{q_1}(Z)} \cdot \|\beta\|_{T^{p_2}_{q_2}(Z)} \lesssim \|\lambda\|_{T^p_q(Z)}$.

The following result can be found in [16, Lemma 3].

Lemma 2.12. *Let* $0 < p < \infty$ *and* $\beta \ge 0$. *There exists* $r_0 \in (0, 1)$ *so that, if* $0 < r < r_0$ *and* $Z = \{z_k\}$ *is an* r*-lattice, then*

$$\int_{\mathbb{T}} \sup_{z \in \Gamma(\xi)} |f(z)|^p \left(1 - |z|^2\right)^\beta |d\xi| \lesssim \int_{\mathbb{T}} \sup_{z_k \in \Gamma(\xi)} |f(z_k)|^p \left(1 - |z_k|^2\right)^\beta |d\xi|,$$

whenever $f \in H(\mathbb{D})$ *is such that the left-hand side is finite.*

We finally make the following observation.

Lemma 2.13. *Let* $\{z_k\}_{k=1}^{\infty}$ *be an* (ρ, r)*-lattice with a small enough* r, *and let* $0 < p < \infty$. *The following statments are equivalent:*

(a) $\lambda = \{\lambda_k\}_{k=1}^{\infty} \in T^p_p(Z)$;

(b) $\left\{ (1 - |z_k|)^{\frac{1}{p}} \lambda_k \right\}_{k=1}^{\infty} \in \ell^p$.

Proof. Assume that $\lambda = \{\lambda_k\} \in T^p_p(Z)$. By the results of [16], there are sequences $\gamma = \{\gamma_k\} \in T^p_\infty(Z)$ and $\beta = \{\beta_k\} \in T^\infty_p(Z)$ such that there holds $\lambda = \gamma \cdot \beta$ with

$\|\gamma\|^p_{T^p_\infty(Z)} \cdot \|\beta\|^p_{T^\infty_p(Z)} \lesssim \|\lambda\|_{T^p_p(Z)}$. We get

$$\sum_k |\lambda_k|^p(1-|z_k| = \sum_k |\gamma_k|^p|\beta_k|^p(1-|z_k|) \lesssim \|\gamma\|^p_{T^p_\infty(Z)} \cdot \|\beta\|^p_{T^\infty_p(Z)} \lesssim \|\lambda\|_{T^p_p(Z)},$$

where the second line is proved in [7].

Conversely, assume that the sequence $\left\{(1-|z_k|)^{\frac{1}{p}}\lambda_k\right\}_{k=1}^\infty$ belongs to ℓ^p. Since $|I(z)| \asymp 1-|z|$ for $z \in \mathbb{D}$, we have

$$\int_{\mathbb{T}} \sum_{\{k: a\in\Gamma(\zeta)\}} |\lambda_k|^p |d\zeta| \lesssim \sum_j |\lambda_k|^p \int_{I(a_k)} |d\zeta| \le C \sum_k |\lambda_k|^p(1-|z_k|).$$

Thus, $\lambda = \{\lambda_k\} \in T^p_p(Z)$, which completes the proof. □

3. Proof of the Main Theorem

In this section, we formulate and prove Theorems 3.1–3.6, from which the Main Theorem follows. The theorems concern three cases of the indices p and q, namely those with either $q = 1$, $p \le q$ or $q < p$. The case $q = 1$ is partially known and the rest of it is a straightforward consequence of the Fubini theorem, whereas the proofs of other cases are more involved and are based on the results presented in the previous section.

Note that the choice of the parameter $\alpha > 0$ was fixed in Section 2.

3.1. The case $q = 1$. In the case $q = 1$ we state the following result, where item (ii) in particular implies part of item (iii) of the Main Theorem; see the proof of Theorem 3.4.

Theorem 3.1. *Let φ be a function belonging to the class $\mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$, and let μ be a finite positive Borel measure on $\mathbb{D}$.*

(i) *Let $0 < p \le 1$. Then, the following statements are equivalent:*
  (a) *$A_\mu\colon A^p_\varphi \to L^1(\mathbb{T})$ is bounded;*
  (b) *$\widetilde{\nu}_t \rho^{2-2/p} \in L^\infty$;*
  (c) *$\widehat{\nu}_\delta \rho^{2-2/p} \in L^\infty$ for some (or any) small enough $\delta \in (0,\alpha]$;*
  (d) *$\left\{\widetilde{\nu}_r(z_k)\,\rho(z_k)^{2-2/p}\right\}_k \in \ell^\infty$ for some (or any) (ρ,r)-lattice $\{z_k\}$ with a small enough $r \in (0,\alpha]$.*

(ii) *Let $1 < p < \infty$. Then, the following statements are equivalent:*
  (a) *$A_\mu\colon A^p_\varphi \to L^1(\mathbb{T})$ is bounded;*
  (b) *$\widetilde{\nu}_t \in L^{\frac{p}{p-1}}$;*
  (c) *$\widehat{\nu}_\delta \in L^{\frac{p}{p-1}}$ for some (or any) small enough $\delta \in (0,\alpha]$;*
  (d) *$\left\{\widehat{\nu}_r(z_k)\,\rho(z_k)^{2-2/p}\right\}_k \in \ell^{\frac{p}{p-1}}$ for some (or any) (ρ,r)-lattice $\{z_k\}$ with $r \in (0,\alpha]$.*

Proof. We only need to prove (a) $\iff$ (b) of (i) and (ii), since the remaining implications in this theorem have been verified in [2, 3, 14]. By Fubini's theorem, we have

$$\begin{aligned}\|A_\mu f\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f(z)| \frac{d\mu(z)}{1-|z|^2} |d\zeta| = \int_{\mathbb{D}} \frac{|f(z)|}{1-|z|^2} \int_{I(z)} |d\zeta|\, d\mu(z)\\ &\asymp \int_{\mathbb{D}} |f(z)|\, d\mu(z) = \int_{\mathbb{D}} |f(z)e^{-\varphi(z)}| e^{\varphi(z)}\, d\mu(z).\end{aligned}$$

Let $d\nu(z) := e^{\varphi(z)}\, d\mu(z)$. Thus, the operator $A_\mu\colon A^p_\varphi \to L^1(\mathbb{T})$ is bounded if and only if ν is a 1-Carleson measure for A^p_φ. Hence, the result follows from Lemmas 2.8 and 2.9. □

3.2. The case $p \le q$. In this case we formulate two theorems, namely Theorems 3.2 and 3.3, which coincide with items (i) and (ii) of the Main Theorem, respectively.

Theorem 3.2. *Let $1 < p \le q < \infty$ and assume that $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$ and that μ is a finite positive Borel measure on $\mathbb{D}$. Then, the operator $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if $\widehat{\mu}_{\delta,1}(z)^{p'} dA(z)$ is a p'/q'-Carleson measure for some (or any) small enough $\delta \in (0,\alpha]$.*

Proof. If $1 < q < \infty$, then, by duality, the area operator $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if there is a positive constant C such that

$$\int_{\mathbb{T}} A_\mu(f)(\zeta) g(\zeta) |d\zeta| \le C \|g\|_{L^{q'}(\mathbb{T})} \|f\|_{A^p_\varphi}, \tag{3.1}$$

for every positive function $g \in L^{q'}(\mathbb{T})$, where q' is the conjugate exponent of q. An application of Fubini's theorem yields

$$\begin{aligned}\int_{\mathbb{T}} A_\mu(f)(\zeta) g(\zeta) |d\zeta| &= \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f(z)| \frac{d\mu(z)}{1-|z|^2} g(\zeta) |d\zeta| \\ &= \int_{\mathbb{D}} \left(\frac{1}{1-|z|^2} \int_{I(z)} g(\zeta)\, d\zeta \right) |f(z)|\, d\mu(z) \\ &= \int_{\mathbb{D}} |f(z) e^{-\varphi(z)}| Tg(z) e^{\varphi(z)}\, d\mu(z).\end{aligned}$$

Let $dv(z) := Tg(z) e^{\varphi(z)}\, d\mu(z)$. In view of (3.1), we see that $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if $\mathrm{Id}\colon A^p_\varphi \to L^1_\varphi(v)$ is bounded, which is equivalent to saying v is a 1-Carleson measure. Using the characterization of 1-Carleson measures in Lemma 2.9, we conclude that $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded, if and only if $\widehat{v}_\delta \in L^{\frac{p}{p-1}}$ and there holds

$$\int_{\mathbb{D}} \left| \int_{D^\delta(z)} Tg(w) \rho(w)^{-2} e^{\varphi(w)}\, d\mu(w) \right|^{p'} dA(z) \le C \|g\|^{p'}_{L^{q'}(\mathbb{T})} \tag{3.2}$$

for all positive $g \in L^{q'}(\mathbb{T})$.

Let us now assume that $\widehat{\mu}_{\delta,1}(z)^{p'} dA(z)$ is a p'/q'-Carleson measure for some (or any) $\delta \in (0,\alpha]$ small enough. We use the facts that $Tg(w) \le C Pg(w)$ and $|Pg(w)| \lesssim |Pg(z)|$ for $w \in D^\delta(z)$ (see Theorem 2.4 and the proof of Theorem 3 in [17]) and Lemma 2.6 to obtain (3.2):

$$\begin{aligned}&\int_{\mathbb{D}} \left| \int_{D^\delta(z)} Tg(w) \rho(w)^{-2} e^{\varphi(w)}\, d\mu(w) \right|^{p'} dA(z) \\ &\le C \int_{\mathbb{D}} |Pg(z)|^{p'} \widehat{\mu}_{\delta,1}(z)^{p'} dA(z) \le C \|g\|^{p'}_{L^{q'}(\mathbb{T})}.\end{aligned}$$

This completes the proof of the "if"-statement. To prove the converse implication, we consider an arc $I \subset \mathbb{T}$ and take $g = \chi_I$ in (3.2) and obtain

$$\int_{S(I)} \left(\int_{D^\delta(z)} |T\chi_I(w)|\, \rho(w)^{-2} e^{\varphi(w)}\, d\mu(w) \right)^{p'} dA(z) \le C |I|^{\frac{p'}{q'}}.$$

Then, the proof is completed by observing that $T\chi_I(w) \geq 1$ for $w \in D^\delta(z)$ and $z \in S(I)$. □

The second assertion of this section reads as follows.

Theorem 3.3. *Assume $p \leq \min\{1,q\}$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$, and let μ be a finite positive Borel measure on $\mathbb{D}$. Then, the area operator $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if for all sufficiently small $r > 0$ there holds*

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|)^{\frac{1-q}{q}}}{\rho(w)^{(2-2p)/p}} \widehat{\mu}_{r,1}(w) < \infty. \tag{3.3}$$

Proof. Assume first that $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded. We fix $w \in \mathbb{D}$ and consider the normalized kernel κ_w (see Section 1) as a test function. By Lemma 2.4, we get

$$\|A_\mu \kappa_w\|_{L^q(\mathbb{T})} \leq C\,\|\kappa_w\|_{A^p_\varphi} \leq C\rho(w)^{2/p-1}, \tag{3.4}$$

for some positive constant C. On the other hand, there is an $r > 0$ (independent of w and ζ) with $D^r(w) \subset \Gamma(\zeta)$ for $\zeta \in \frac{1}{2}I(w)$. Here we use the notation sI with $s > 0$ to denote the arc with the same center as the arc I and length $s|I|$. Therefore, using the fact that $|\kappa_w(z)| \asymp \frac{1}{\rho(w)} e^{\varphi(z)}$ for $z \in D^r(w)$ (see Lemma 2.3) one obtains

$$\begin{aligned}
\|A_\mu(\kappa_w)\|^q_{L^q(\mathbb{T})} &\geq \int_{\frac{1}{2}I(w)} (A_\mu\kappa_w(\zeta))^q\,|d\zeta| \\
&= \int_{\frac{1}{2}I(w)} \left(\int_{\Gamma(\zeta)} |\kappa_w(z)|\,\frac{d\mu(z)}{1-|z|^2}\right)^q |d\zeta| \geq \int_{\frac{1}{2}I(w)} \left(\int_{D^r(w)} |\kappa_w(z)|\,\frac{d\mu(z)}{1-|z|^2}\right)^q |d\zeta| \\
&\geq C|I(w)| \left(\int_{D^r(w)} \frac{1}{\rho(w)} e^{\varphi(z)} \frac{d\mu(z)}{1-|z|^2}\right)^q \\
&\geq C(1-|w|)^{1-q} \left(\int_{D^r(w)} \frac{1}{\rho(w)} e^{\varphi(z)}\,d\mu(z)\right)^q.
\end{aligned}$$

Combining this with (3.4) yields

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|)^{\frac{1-q}{q}}}{\rho(w)^{(2-2p)/p}} \widehat{\mu}_{r,1}(w) < \infty.$$

Conversely, assume that (3.3) holds. Let $r \in (0,\alpha]$, and let $\{z_k\}_{k=1}^\infty$ be a (ρ,r)-lattice on $\mathbb{D}$. We adopt the notation $\widetilde{D}^r(z_k) = \bigcup_{z\in D^r(z_k)} D^r(z)$. Taking into account Lemma 2.2 we obtain

$$\begin{aligned}
A_\mu f(\zeta) &\lesssim \int_{\Gamma(\zeta)} \left(\frac{e^{p\varphi(z)}}{\rho(z)^2} \int_{D^r(z)} |f(\xi)|^p e^{-p\varphi(\xi)}\,dA(\xi)\right)^{1/p} \frac{d\mu(z)}{1-|z|} \\
&\lesssim \sum_{k\in\mathbb{N}(\zeta)} \int_{D^r(z_k)} \frac{e^{\varphi(z)}}{\rho(z)^{2/p}} \left(\int_{D^r(z)} |f(\xi)|^p e^{-p\varphi(\xi)}\,dA(\xi)\right)^{1/p} \frac{d\mu(z)}{1-|z|} \\
&\lesssim \sum_{k\in\mathbb{N}(\zeta)} \left(\int_{\widetilde{D}^r(z_k)} |f(\xi)|^p e^{-p\varphi(\xi)}\,dA(\xi)\right)^{1/p} \frac{(1-|z_k|)^{-1}}{\rho(z_k)^{2/p}} \int_{D^r(z_k)} e^{\varphi(z)}\,d\mu(z),
\end{aligned}$$

where we denoted $\mathbb{N}(\zeta)=\{k\in\mathbb{N}\colon D^r(z_k)\cap\Gamma(\zeta)\neq\emptyset\}$ and also used that $\rho(z)\asymp\rho(z_k)$ for $z\in D^r(z_k)$ in the last inequality. This, together with the assumption (3.3) yields

$$A_\mu f(\zeta)\le C\sum_{k\in\mathbb{N}(\zeta)}\left(\int_{\widetilde{D}^r(z_k)}|f(\xi)|^pe^{-p\varphi(\xi)}\,dA(\xi)\right)^{1/p}(1-|z_k|)^{-1/q}.$$

Since $0<p\le 1$ and there holds $1-|z_k|\asymp 1-|\xi|$ for $\xi\in\widetilde{D}^r(z_k)$, we get

$$\begin{aligned}A_\mu f(\zeta)^p&\le C\sum_{k\in\mathbb{N}(\zeta)}\int_{\widetilde{D}^r(z_k)}|f(\xi)|^p\,(1-|\xi|)^{-p/q}\,e^{-p\varphi(\xi)}\,dA(\xi)\\&\le C\int_{\widetilde{\Gamma}(\zeta)}|f(\xi)|^p(1-|\xi|)^{-p/q}e^{-p\varphi(\xi)}\,dA(\xi),\end{aligned}$$

where $\widetilde{\Gamma}(\zeta)$ is a Stolz angle with vertex at ζ with a bigger aperture than $\Gamma(\zeta)$. Thus, by Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}\|A_\mu f\|_{L^q}^q&=\int_{\mathbb{T}}(A_\mu f(\zeta)^p)^{q/p}\,|d\zeta|\\&\le C\int_{\mathbb{T}}\left(\int_{\widetilde{\Gamma}(\zeta)}|f(\xi)|^p(1-|\xi|)^{-p/q}e^{-p\varphi(\xi)}\,dA(\xi)\right)^{q/p}|d\zeta|\\&\le C\int_{\mathbb{T}}\left(\int_{\widetilde{\Gamma}(\zeta)}|f(\xi)|^pe^{-p\varphi(\xi)}\,dA(\xi)\right)^{(q-p)/p}\left(\int_{\widetilde{\Gamma}(\zeta)}|f(\xi)|^pe^{-p\varphi(\xi)}\frac{dA(\xi)}{1-|\xi|)}\right)|d\zeta|\\&\le C\|f\|_{A^p_\varphi}^{q-p}\int_{\mathbb{T}}\int_{\widetilde{\Gamma}(\zeta)}|f(\xi)|^pe^{-p\varphi(\xi)}\frac{dA(\xi)}{1-|\xi|}|d\zeta|\\&=C\|f\|_{A^p_\varphi}^{q-p}\int_{\mathbb{D}}|f(\xi)|^pe^{-p\varphi(\xi)}\left(\int_{\mathbb{T}}\chi_{\widetilde{\Gamma}(\zeta)}(\xi)|d\zeta|\right)\frac{dA(\xi)}{1-|\xi|}\\&\le C\|f\|_{A^p_\varphi}^q,\end{aligned}$$

where the last inequality is due to the fact that $\int_{\mathbb{T}}\chi_{\widetilde{\Gamma}(\zeta)}(\xi)|d\zeta|\asymp 1-|\xi|$. This finishes the proof. □

3.3. The case $q<p$. In this final section we present Theorems 3.4, 3.5 and 3.6, which correspond to items (iii), (iv) and (v) of the Main Theorem, respectively, and thus complete its proof.

Theorem 3.4. *Let $1\le q<p<\infty$ and $\varphi\in\mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}}\asymp\rho\in\mathcal{L}_0$ and let μ be a finite positive Borel measure on $\mathbb{D}$. Then $A_\mu\colon A^p_\varphi\to L^q(\mathbb{T})$ is bounded if and only if for some (or any) sufficiently small $\delta\in(0,\alpha]$ there holds*

$$G(\zeta)=\int_{\Gamma(\zeta)}\frac{\widehat{\mu}_{\delta,1}(z)^{p'}}{1-|z|}\,dA(z)\in L^{\frac{q(p-1)}{p-q}}(\mathbb{T}).\tag{3.5}$$

Proof. Observe that, by Fubini's theorem, the condition $G\in L^1(\mathbb{T})$ is equivalent to the relation $\widehat{\nu}_\delta(z)=\widehat{\mu}_{\delta,1}(z)\in L^{p/(p-1)}(\mathbb{D})$. Thus, the case $q=1$ is exactly (ii)(c) of Theorem 3.1, and it suffices to consider the case $1<q<\infty$. In the same way as in the proof of Theorem 3.2, we see that $A_\mu\colon A^p_\varphi\to L^q(\mathbb{T})$ is bounded if and only if $\widehat{v}_\delta\in L^{\frac{p}{p-1}}$ and there holds

$$\int_{\mathbb{D}}\left|\int_{D^\delta(z)}Tg(w)\rho(w)^{-2}e^{\varphi(w)}\,d\mu(w)\right|^{p'}dA(z)\le C\|g\|_{L^{q'}(\mathbb{T})}^{p'},\tag{3.6}$$

for all positive $g \in L^{q'}(\mathbb{T})$.

Let us assume (3.5) holds. Recalling that $|Tg(w)| \lesssim |Pg(w)|$ and $|Pg(w)| \lesssim |Pg(z)|$ for $w \in D^\delta(z)$ yields

$$\begin{aligned}(3.7)\qquad &\int_{\mathbb{D}} \left| \int_{D^\delta(z)} Tg(w)\rho(w)^{-2} e^{\varphi(w)}\, d\mu(w) \right|^{p'} dA(z) \\ &\leq C \int_{\mathbb{D}} |Pg(z)|^{p'} \widehat{\mu}_{\delta,1}(z)^{p'} dA(z).\end{aligned}$$

Denoting $dm := \widehat{\mu}_{\delta,1}(z)^{p'} dA(z)$, we observe that the assumption (3.5) is equivalent with

$$A_\mu 1(\zeta) = \int_{\Gamma(\zeta)} \frac{dm(z)}{1-|z|} \in L^{\frac{q'}{q'-p'}}(\mathbb{T}),$$

since $\frac{q'}{q'-p'} = \frac{q(p-1)}{p-q}$ with $p' < q'$. Hence, Lemma 2.7 gives that $\mathrm{Id}\colon H^{q'} \to L^{p'}(m)$ is bounded. We obtain

$$\int_{\mathbb{D}} |Pg(z)|^{p'} \widehat{\mu}_{\delta,1}(z)^{p'} dA(z) = \|Pg\|^{p'}_{L^{p'}(m)} \leq C \|g\|^{p'}_{L^{q'}(\mathbb{T})}$$

and combining this with (3.7) shows that (3.6) holds. Therefore, the a area operator $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded.

For the converse implication, we denote by $u^*(\zeta) = \sup_{z\in\Gamma(\zeta)} |u(z)|$ the non-tangential maximal function of u. We integrate both sides of the obvious inequality $Ph(z) \leq (Ph)^*(w)$ for $w \in I(z)$ with respect to w on $I(z)$ to obtain

$$Ph(z) \asymp \frac{Ph(z)}{1-|z|} \int_{I(z)} dw \lesssim \frac{1}{1-|z|} \int_{I(z)} (Ph)^*(w)\, dw = T((Ph)^*)(z).$$

Hence, applying (3.6) with $g = (Ph)^*$ yields

$$\begin{aligned}&\int_{\mathbb{D}} \left| \int_{D^\delta(z)} |Ph(w)| \rho(w)^{-2} e^{\varphi(w)}\, d\mu(w) \right|^{p'} dA(z) \\ &\lesssim \int_{\mathbb{D}} \left| \int_{D^\delta(z)} |Tg(w)| \rho(w)^{-2} e^{\varphi(w)}\, d\mu(w) \right|^{p'} dA(z) \leq C \|h\|^{p'}_{L^{q'}(\mathbb{T})}.\end{aligned}$$

Noting that $|Ph(w)| \asymp |Ph(z)|$ for $w \in D^\delta(z)$, we get

$$\int_{\mathbb{D}} |Ph(z)|^{p'} \widehat{\mu}_{\delta,1}(z)^{p'} dA(z) \leq C \|h\|^{p'}_{L^{q'}(\mathbb{T})}.$$

Thus, $\mathrm{Id}\colon H^{q'} \to L^{p'}(m)$ is bounded and by Lemma 2.7 we get $G(\zeta) \in L^{\frac{q(p-1)}{p-q}}(\mathbb{T})$, which completes the proof. □

Theorem 3.5. *Let $0 < q < p \leq 1$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$ and μ be a finite positive Borel measure on $\mathbb{D}$. Then $A_\mu\colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if for any sufficiently small $r > 0$, the function*

$$G^\mu_{p,q}(\zeta) := \sup_{w\in\widetilde{\Gamma}(\zeta)} \frac{(1-|w|)^{\frac{1-p}{p}}}{\rho(w)^{(2-2p)/p}} \widehat{\mu}_{r,1}(w), \quad \zeta \in \mathbb{T},$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{T})$, where $\widetilde{\Gamma}(\zeta)$ is some Stolz angle with vertex at ζ with a bigger aperture than that of $\Gamma(\zeta)$.

Proof. We first prove the sufficiency. Let $r \in (0,\alpha]$ and let $\{z_k\}_{k=1}^{\infty}$ be a (ρ,r)-lattice on $\mathbb{D}$. We consider first

$$B^{\mu}_{p,q}(w) = \frac{(1-|w|)^{\frac{1-p}{p}}}{\rho(w)^{(2-2p)/p}} \widehat{\mu}_{r,1}(w), \quad w \in \mathbb{D}.$$

By Lemma 2.2, we have

$$\begin{aligned} |A_\mu(f)(\zeta)| &\le \int_{\Gamma(\zeta)} |f(z)| \frac{d\mu(z)}{1-|z|} \\ &\lesssim \int_{\Gamma(\zeta)} \left(\frac{1}{\rho(z)^2} \int_{D^r(z)} |f(\xi)| e^{-\varphi(\xi)}\, dA(\xi) \right) e^{\varphi(z)} \frac{d\mu(z)}{1-|z|}. \end{aligned}$$

By Fubini's theorem, (a) of Lemma 2.1, (2.3) and the fact that $(1-|z|) \asymp (1-|\xi|)$, for $\xi \in D^r(z)$, we get

$$\begin{aligned} (3.8) \qquad |A_\mu(f)(\zeta)| &\lesssim \int_{\Gamma(\zeta)} |f(\xi)| e^{-\varphi(\xi)} \left(\frac{(1-|\xi|)^{-1}}{\rho(\xi)^2} \int_{D^{Br}(\xi)} e^{\varphi(z)}\, d\mu(z) \right) dA(\xi) \\ &= \int_{\Gamma(\zeta)} |f(\xi)| e^{-\varphi(\xi)} \left(\frac{(1-|\xi|)^{-1/p}}{\rho(\xi)^{2(1-\frac{1}{p})}} B^{\mu}_{p,q}(\xi) \right) dA(\xi) \\ &\le \sup_{\xi \in \Gamma(\zeta)} B^{\mu}_{p,q}(\xi) \int_{\Gamma(\zeta)} |f(\xi)| e^{-\varphi(\xi)} \frac{(1-|\xi|)^{-1/p}}{\rho(\xi)^{2(1-\frac{1}{p})}}\, dA(\xi) \\ &= G^{\mu}_{p,q}(\zeta) \int_{\Gamma(\zeta)} |f(\xi)| e^{-\varphi(\xi)} \frac{(1-|\xi|)^{-1/p}}{\rho(\xi)^{2(1-\frac{1}{p})}}\, dA(\xi). \end{aligned}$$

By Hölder's inequality, we obtain

$$(3.9) \qquad \int_{\mathbb{T}} |A_\mu(f)(\zeta)|^q |d\zeta| \lesssim \left(\int_{\mathbb{T}} G^{\mu}_{p,q}(\zeta)^{pq/(p-q)} |d\zeta| \right)^{(p-q)/p} \left(\int_{\mathbb{T}} (II)^p\, |d\zeta| \right)^{q/p},$$

where

$$II(\zeta) = \int_{\Gamma(\zeta)} |f(\xi)| e^{-\varphi(\xi)} \frac{(1-|\xi|)^{-1/p}}{\rho(\xi)^{2(1-\frac{1}{p})}}\, dA(\xi)$$

On the other hand, Lemma 2.2 and (a) of Lemma 2.1 yield

$$\begin{aligned} II(\zeta) &\lesssim \sum_{k \in \mathbb{N}(\zeta)} \int_{D^r(z_k)} \frac{(1-|\xi|)^{-1/p}}{\rho(\xi)^{2(1-\frac{1}{p})}} \left(\frac{1}{\rho(\xi)^2} \int_{D^r(\xi)} |f(z)|^p e^{-p\varphi(z)}\, dA(z) \right)^{1/p} dA(\xi) \\ &\lesssim \sum_{k \in \mathbb{N}(\zeta)} \int_{D^r(z_k)} \frac{(1-|\xi|)^{-1/p}}{\rho(\xi)^2} \left(\int_{D^r(\xi)} |f(z)|^p e^{-p\varphi(z)}\, dA(z) \right)^{1/p} dA(\xi) \\ &\lesssim \sum_{k \in \mathbb{N}(\zeta)} (1-|z_k|)^{-1/p} \left(\int_{D^{Br}(\xi)} |f(z)|^p e^{-p\varphi(z)}\, dA(z) \right)^{1/p}. \end{aligned}$$

where $\mathbb{N}(\zeta) = \{k \in \mathbb{N} \colon D^r(z_k) \cap \Gamma(\zeta) \neq \emptyset\}$. Since $p \le 1$, we obtain

$$\begin{aligned} II(\zeta)^p &\lesssim \sum_{k \in \mathbb{N}(\zeta)} (1-|z_k|)^{-1/p} \left(\int_{D^{Br}(\xi)} |f(z)|^p e^{-p\varphi(z)}\, dA(z) \right)^{1/p} \\ &\lesssim \int_{\Gamma(\zeta)} |f(z)|^p e^{-p\varphi(z)} \frac{dA(z)}{1-|z|}. \end{aligned}$$

Then, applying Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{T}} II(\zeta)^p |d\zeta| &\lesssim \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f(z)|^p e^{-p\varphi(z)} \frac{dA(z)}{1-|z|} |d\zeta| \\ &\leq \int_{\mathbb{D}} |f(z)|^p e^{-p\varphi(z)} \frac{dA(z)}{1-|z|} \int_{I(z)} |d\zeta|,\end{aligned}$$

where $I(z) = \{\zeta \in \partial\mathbb{D} \colon z \in \Gamma(\zeta)\}$. Since $|I(z)| \asymp (1-|z|)$, we get

$$\int_{\mathbb{T}} II(\zeta)^p |d\zeta| \lesssim \|f\|^p_{A^p_\varphi}.$$

By inserting this into (3.9), we obtain

$$\int_{\mathbb{T}} |A_\mu(f)(\zeta)|^q |d\zeta| \lesssim \|G^\mu_{p,q}\|^q_{L^{pq/(p-q)}(\mathbb{T})} \, \|f\|^q_{A^p_\varphi}.$$

Thus,

$$\|A_\mu(f)\|_{L^q(\mathbb{T})} \lesssim \|G^\mu_{p,q}\|_{L^{pq/(p-q)}(\mathbb{T})} \, \|f\|_{A^p_\varphi}$$

so that A_μ is bounded.

Next, we prove the necessity. The proof follows the idea in [15]. Let $\{z_k\}_{k=1}^\infty$ be a (ρ, r)-lattice with a small enough $r \in (0, \alpha]$. The test function F_t under concern is defined for $z \in \mathbb{D}$ as

$$F_t(z) = \sum_k (1-|z_k|)^{\frac{1}{p}} \lambda_k r_k(t) \kappa_{z_k}(z) \rho(z_k)^{1-\frac{2}{p}},$$

where $\lambda = \{\lambda_k\} \in T^p_p(Z)$, and $r_k \colon [0,1] \to \{0, \pm 1\}$ are the Rademacher functions. For $t \in [0,1]$, Lemmas 2.5 and 2.13 show that $F_t \in A^p_\varphi$ and

$$\|F_t\|_{A^p_\varphi} \leq C \|\lambda\|_{T^p_p(Z)}.$$

On the other hand, it follows from the boundedness of $A_\mu \colon A^p_\varphi \to L^q(\mathbb{T})$ that

$$\begin{aligned}&\int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} \left| \sum_k (1-|z_k|)^{\frac{1}{p}} \lambda_k r_k(t) \kappa_{z_k}(z) \rho(z_k)^{1-\frac{2}{p}} \right| \frac{d\mu(z)}{1-|z|^2} \right)^q |d\zeta| \\ &= \int_{\mathbb{T}} |A_\mu F_t(\zeta)|^q \, |d\zeta| \lesssim \|A_\mu\|^q_{A^p_\varphi \to L^q(\mathbb{T})} \cdot \|\lambda\|^q_{T^p_p(Z)}.\end{aligned}$$

Integrating with respect to t in $[0,1]$ and using the notation

$$\|f\|_\zeta = \int_{\Gamma(\zeta)} |f(z)| \frac{d\mu(z)}{1-|z|^2}$$

for $f \in H(\mathbb{D})$, we get

$$\int_0^1 \int_{\mathbb{T}} \left\| \sum_k (1-|z_k|)^{\frac{1}{p}} \lambda_k r_k(t) \rho(z_k)^{1-\frac{2}{p}} \kappa_{z_k}(\cdot) \right\|^q_\zeta d\zeta \, dt \lesssim \|A_\mu\|^q_{A^p_\varphi \to L^q(\mathbb{T})} \cdot \|\lambda\|^q_{T^p_p(Z)}.$$

By (2.2) and (2.1), we obtain

$$\int_{\mathbb{T}}\left[\int_{\Gamma(\zeta)}\left(\sum_k (1-|z_k|)^{\frac{2}{p}}|\lambda_k|^2|\kappa_{z_k}(z)|^2\rho(z_k)^{2-\frac{4}{p}}\right)^{1/2}\frac{d\mu(z)}{1-|z|}\right]^q d\zeta$$
$$\lesssim \int_{\mathbb{T}}\left(\int_0^1\left\|\sum_k (1-|z_k|)^{\frac{1}{p}}\lambda_k r_k(t)\rho(z_k)^{1-\frac{2}{p}}\kappa_{z_k}(\cdot)\right\|_{\zeta} dt\right)^q d\zeta$$
$$\lesssim \|A_\mu\|^q_{A^p_\varphi\to L^q(\mathbb{T})}\cdot\|\lambda\|^q_{T^p_p(Z)}.$$

On the other hand, there exists a $\tau>1$ such that $D^r(z)\subset\Gamma(\zeta)$ for $z\in\Gamma_\tau(\zeta)$. Thus, Lemma 2.3 yields

$$\int_{\Gamma(\zeta)}\left(\sum_k (1-|z_k|)^{\frac{2}{p}}|\lambda_k|^2|\kappa_{z_k}(z)|^2\rho(z_k)^{2-\frac{4}{p}}\right)^{1/2}\frac{d\mu(z)}{1-|z|}$$
$$\geq \sum_{\{j:z_j\in\Gamma_\tau(\zeta)\}}\int_{\Gamma(\zeta)\cap D^r(z_j)}\left(\sum_k (1-|z_k|)^{\frac{2}{p}}|\lambda_k|^2|\kappa_{z_k}(z)|^2\rho(z_k)^{2-\frac{4}{p}}\right)^{1/2}\frac{d\mu(z)}{1-|z|}$$
$$\geq \sum_{\{j:z_j\in\Gamma_\tau(\zeta)\}}(1-|z_j|)^{\frac{1}{p}}|\lambda_j|\int_{D^r(z_j)}|\kappa_{z_k}(z)|\rho(z_k)^{1-\frac{2}{p}}\frac{d\mu(z)}{1-|z|}$$
$$\gtrsim \sum_{\{j:z_j\in\Gamma_\tau(\zeta)\}}|\lambda_j|\frac{(1-|z_j|)^{\frac{1-p}{p}}}{\rho(z_j)^{2/p}}\int_{D^r(z_j)}e^{\varphi(z)}\,d\mu(z).$$

Therefore, we have

$$\int_{\mathbb{T}}\left(\sum_{\{j:z_j\in\Gamma_\tau(\zeta)\}}|\lambda_j|\frac{(1-|z_j|)^{\frac{1-p}{p}}}{\rho(z_j)^{2/p}}\int_{D^r(z_j)}e^{\varphi(z)}\,d\mu(z)\right)^q|d\zeta| \tag{3.10}$$
$$\lesssim \|A_\mu\|^q_{A^p_\varphi\to L^q(\mathbb{T})}\cdot\|\lambda\|^q_{T^p_p(Z)}.$$

To prove $G^\mu_{p,q}\in L^{\frac{pq}{(p-q)}}(\mathbb{T})$, by Lemma 2.12, it is sufficient to show that

$$K^\mu_{p,q}:=\int_{\mathbb{T}}\sup_{z_k\in\widetilde{\Gamma}(\zeta)}\frac{(1-|z_k|)^{\frac{q(1-p)}{p-q}}}{\rho(z_k)^{2q/p-q}}\left(\int_{D^r(z_k)}e^{\varphi(z)}\,d\mu(z)\right)^{\frac{pq}{p-q}}|d\zeta|<\infty. \tag{3.11}$$

We write, for all $k\in\mathbb{N}$,

$$\nu_k=\frac{(1-|z_k|)^{\frac{q(1-p)}{p}}}{\rho(z_k)^{\frac{2q}{p}}}\left(\int_{D^r(z_k)}e^{\varphi(z)}\,d\mu(z)\right)^q.$$

Then, (3.11) holds if and only if the sequence $\nu=\{\nu_k\}$ belongs to the tent sequence space $T_\infty^{\frac{p}{p-q}}(Z)$. For $t>1$, this is equivalent to the statement $\nu^{1/t}:=\left\{\nu_k^{1/t}\right\}\in T_\infty^{\frac{pt}{p-q}}(Z)$. Choose $t>1$ such that $pt>1$ and write

$$\frac{t-q}{t}+\frac{q}{pt}=\frac{1}{\varrho},$$

where $0<\varrho\leq 1$ due to $0<q<p\leq 1$. By Lemma 2.10 and Lemma 2.11, we have

$$T_\infty^{\frac{pt}{p-q}}(Z)=\left(T_\varrho^{\left(\frac{pt}{p-q}\right)'}(Z)\right)^*=\left(T^{t_1'}_{\frac{t}{t-q}}(Z)\cdot T^{\frac{pt}{q}}_{\frac{pt}{q}}(Z)\right)^*.$$

Take any $\eta = \{\eta_k\} \in T_\varrho^{\left(\frac{pt}{p-q}\right)'}(Z)$, and factor it as $\eta_k = \tau_k \lambda_k^{q/t}$, with $\tau = \{\tau_k\} \in T^{t'}_{\frac{t}{t-q}}(Z)$ and $\lambda = \{\lambda_k\} \in T_p^p(Z)$. Using Fubini's theorem, Hölder's inequality twice, (3.10) and Lemma 2.11 and denoting $\mathbb{N}(\zeta) = \{k \in \mathbb{N} \colon z_k \in \widetilde{\Gamma}(\zeta)\}$, we obtain

$$\begin{aligned}
(3.12)\qquad \sum_{k\in\mathbb{N}(\zeta)} |\eta_k| \left|\nu_k^{1/t}\right| (1-|z_k|) &\asymp \int_{\mathbb{T}} \left(\sum_{k\in\mathbb{N}(\zeta)} \tau_k \lambda_k^{q/t} \nu_k^{1/t} \right) |d\zeta| \\
&\le \int_{\mathbb{T}} \left(\sum_{k\in\mathbb{N}(\zeta)} \tau_k^{(t/q)'} \right)^{\frac{t-q}{t}} \left(\sum_{k\in\mathbb{N}(\zeta)} \lambda_k \nu_k^{1/q} \right)^{\frac{q}{t}} |d\zeta| \\
&\lesssim \|\tau\|_{T^{t'}_{\frac{t}{t-q}}} \left[\int_{\mathbb{T}} \left(\sum_{k\in\mathbb{N}(\zeta)} \lambda_k \nu_k^{1/q} \right)^{q} |d\zeta| \right]^{1/t} \\
&\lesssim \|\tau\|_{T^{t'}_{\frac{t}{t-q}}} \|A_\mu\|^{q/t}_{A^p_\varphi \to L^q(\mathbb{T})} \cdot \|\lambda\|^{q/t}_{T_p^p(Z)} \lesssim \|\eta\|_{T_\varrho^{\left(\frac{pt}{p-q}\right)'}} \cdot \|A_\mu\|^{q/t}_{A^p_\varphi \to L^q(\mathbb{T})}.
\end{aligned}$$

By duality, we get $\nu^{1/t} \in T_\infty^{\frac{pt}{p-q}}(Z)$. The proof is complete. □

Our last theorem reads as follows.

Theorem 3.6. *Let $0 < q < 1 \le p$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \asymp \rho \in \mathcal{L}_0$, and let μ be a finite positive Borel measure on $\mathbb{D}$. Then, $A_\mu \colon A^p_\varphi \to L^q(\mathbb{T})$ is bounded if and only if for all sufficiently small $r > 0$, the function*

$$M^\mu_{p,q}(\zeta) := \int_{\widetilde{\Gamma}(\zeta)} \frac{(1-|w|)^{\frac{1-p}{p}}}{\rho(w)^{2/p}} \widehat{\mu}_{r,1}(w)\, dA(w), \quad \zeta \in \mathbb{T},$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{T})$.

Proof. We first discuss the "if" part. Using (2.3) and denoting $\mathbb{N}(\zeta) = \{k \in \mathbb{N} \colon D^r(z_k) \cap \widetilde{\Gamma}(\zeta) \neq \emptyset\}$, we get

$$\begin{aligned}
M^\mu_{p,q}(\zeta) &\le \sum_{k\in\mathbb{N}(\zeta)} \int_{D^r(z_k)} \frac{(1-|w|)^{\frac{1-p}{p}}}{\rho(w)^{(2+2p)/p}} \int_{D^r(w)} e^{\varphi(z)}\, d\mu(z)\, dA(w) \\
&\le \sum_{k\in\mathbb{N}(\zeta)} \frac{(1-|z_k|)^{\frac{1-p}{p}}}{\rho(z_k)^{(2+2p)/p}} \rho(z_k)^2 \int_{D^{Br}(z_k)} e^{\varphi(z)}\, d\mu(z) \\
&= \sum_{k\in\mathbb{N}(\zeta)} \frac{(1-|z_k|)^{\frac{1-p}{p}}}{\rho(z_k)^{2/p}} \int_{D^{Br}(z_k)} e^{\varphi(z)}\, d\mu(z).
\end{aligned}$$

Write, for all $k \in \mathbb{N}$,

$$\nu_k = \frac{(1-|z_k|)^{\frac{q(1-p)}{p}}}{\rho(z_k)^{\frac{2q}{p}}} \left(\int_{D^{Br}(z_k)} e^{\varphi(z)}\, d\mu(z) \right)^q.$$

It is clear that we only need prove $\nu = \{\nu_k\} \in T_{\frac{1}{q}}^{\frac{p}{p-q}}(Z)$. For $t > 1$, this is equivalent to the statement $\nu^{1/t} := \left\{\nu_k^{1/t}\right\} \in T_{\frac{t}{q}}^{\frac{pt}{p-q}}(Z)$. By Lemma 2.10 and Lemma 2.11, we have

$$T^{\frac{pt}{p-q}}_{\frac{t}{q}}(Z) = \left(T^{\left(\frac{pt}{p-q}\right)'}_{(\frac{t}{q})'}(Z)\right)^* = \left(T^{t'}_{\varrho}(Z)\cdot T^{\frac{pt}{q}}_{\frac{pt}{q}}(Z)\right)^*,$$

since

$$\frac{1}{\varrho} + \frac{q}{pt} = \frac{1}{(\frac{t}{q})'}.$$

Take any $\eta = \{\eta_k\} \in T^{\left(\frac{pt}{p-q}\right)'}_{(\frac{t}{q})'}(Z)$, and factor it as $\eta_k = \tau_k \lambda_k^{q/t}$, with $\tau = \{\tau_k\} \in T^{t'}_{\varrho}(Z)$ and $\lambda = \{\lambda_k\} \in T^p_p(Z)$. We obtain $\nu^{1/t} \in T^{\frac{pt}{p-q}}_{\frac{t}{q}}(Z)$ similarly to (3.12) in the proof of Theorem 3.5.

We finally prove the "only if" part, by arguing in the same way as in the sufficiency proof of Theorem 3.5 and denoting $\mathbb{N}(\zeta) = \{k \in \mathbb{N} \colon D^r(z_k) \cap \Gamma(\zeta) \neq \emptyset\}$:

$$\begin{aligned}
&\sum_{k\in\mathbb{N}(\zeta)} \frac{(1-|z_k)^{\frac{1-p}{p}}}{\rho\left(z_k\right)^{2/p}} \int_{D^r(z_k)} e^{\varphi(z)}\, d\mu(z) \\
&= \sum_{k\in\mathbb{N}(\zeta)} \frac{(1-|z_k)^{\frac{1-p}{p}}}{\rho\left(z_k\right)^{(2+2p)/p}} \rho\left(z_k\right)^2 \int_{D^r(z_k)} e^{\varphi(z)}\, d\mu(z) \\
&\lesssim \sum_{k\in\mathbb{N}(\zeta)} \int_{D^r(z_k)} \frac{(1-|w|)^{\frac{1-p}{p}}}{\rho(w)^{(2+2p)/p}} \int_{D^{Br}(w)} e^{\varphi(z)}\, d\mu(z)\, dA(w) \lesssim M^{\mu}_{p,q}(w).
\end{aligned}$$

This gives the desired results and completes the proof. □

Remark 3.7. From our main result one can deduce the following two statements:

- For $1 \leq q < p = \infty$, $A_\mu \colon A^\infty_\varphi \to L^q(\mathbb{T})$ is bounded if and only if

$$G(\zeta) := \int_{\Gamma(\zeta)} \frac{\widehat{\mu}_{\delta,1}(z)}{1-|z|}\, dA(z) \in L^q(\mathbb{T})$$

  for some (or any) sufficiently small $\delta \in (0, \alpha]$.
- For $q \leq 1 < p = \infty$, the operator $A_\mu \colon A^\infty_\varphi \to L^q(\mathbb{T})$ is bounded if and only if

$$M^{\mu}_{\infty,q}(\zeta) := \int_{\widetilde{\Gamma}(\zeta)} \frac{\widehat{\mu}_{\delta,1}(z)}{1-|z|}\, dA(z)$$

  belongs to $L^q(\mathbb{T})$ for every sufficiently small $r > 0$.

We thus observe that the sufficient and necessary conditions are almost the same in both cases $1 \leq q < p = \infty$ or $q \leq 1 < p = \infty$.

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